The Role of Network Connectivity in Distributed k-Agreement Protocols ⋆

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Abstract

Given a network of agents, we say that the agents achieve a k-*agreement* when their state variables converge to a point that corresponds to the projection of the agents' states onto a k-dimensional linear subspace. The k-agreement problem generalizes the classical consensus problem; unlike in consensus, where the agents' states must asymptotically coincide, in k-agreement the agents reach an agreement in a generalized sense (within a linear subspace, where the states do not necessarily coincide). In this paper, we investigate which interaction topologies enable a network of agents to reach an agreement on a prescribed k-dimensional subspace through local coordination algorithms. We show that achieving k-agreement requires communication over highly connected graphs; specifically, the number of edges in the interaction graph must grow linearly with the dimension k of the agreement subspace. Our characterization reveals that the presence of cycles in the communication graph (particularly, independent families of cycles) constitutes the fundamental structural feature enabling the agents to achieve k-agreement. We also investigate the use of common graph topologies, such as path and circulant graphs, for k-agreement, deriving insights into the relationship between the subspace dimension k and the required network connectivity. The effectiveness of the proposed framework is demonstrated through simulations in robotic formation control problems.

Key words: Multi-agent systems, Decentralized and distributed control, Networked robotic systems, Cooperative systems

1 Introduction

Distributed coordination algorithms play a fundamental role in several network synchronization problems, including rendezvous, distributed optimization, distributed computation and sensing, federated learning, and much more. A common objective in network coordination problems is that of making a group of agents agree on a common quantity. This problem is often referred to as consensus [28] and a vast body of literature has been developed on it—see the (non-exhaustive list) representative works [6, 28, 33]. In other cases, it is instead of interest to make the agents agree in a generalized sense: rather than on a common quantity, one may be interested in ensuring that the agents' states converge to a vector that belongs to a certain set (e.g., a vector space). When the agreement set is a linear subspace and the coordination protocol is linear, the problem is referred to as k-*dimensional agreement* (or simply k-*agreement*) [5]. This problem was investigated in our recent work [5], where we provided algebraic characterizations of the corresponding algorithms and studied their design to optimize the rate of convergence. One key finding of [5] is that, in general, k-agreement protocols require interaction graphs with higher connectivity than those used for simpler coordination algorithms, such as average consensus [28]. In this paper, we seek to provide answers to the following question: *what topological properties of the interaction graph ensure that a set of agents can reach a k-agreement?* Our findings in this paper extend [5] in several directions: (i) we derive necessary conditions on the topology of the communication graph to enable a k-agreement; (ii) we show that k- agreement is possible when the interaction topology incorporates a sufficient number of independent cycles; and (iii) we provide insights into the design of graphs that support k-agreement protocols.

An important application of k-agreement problems is robotic formation control [8, 26], where achieving a certain configuration for the team amounts to ensuring that the vector of agents' positions belongs to a certain set. In this work, we explore this application and we illustrate how k-agreement provides a natural framework to specify constraints to be satisfied by the team of robots at convergence.

Related work. The k-agreement problem is fundamentally related to distributed consensus, a topic that has

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received extensive attention in the literature. Over the years, numerous aspects of consensus algorithms have been studied. Although the list is necessarily incomplete, key areas of investigation include: necessary and sufficient conditions for achieving consensus [9, 18, 19, 28, 32, 39]; the impact of communication time delays [9]; consensus protocols incorporating linear objective maps [13]; distributed optimization approaches based on the alternating direction method of multipliers (ADMM) [7,16,36]; strategies for handling quantized measurements [20]; analysis of convergence rates [29, 41]; and robustness to disturbances and model uncertainties [15, 21], among many others. These contributions form the foundation for a wide range of applications in networked systems, multi-agent coordination, and distributed control. The k-agreement problem is closely related to constrained consensus [24, 25] and distributed optimization with global constraints [40]. However, unlike these formulations, k-agreement imposes constraints that must be satisfied not only during transients but also in the asymptotic regime. Moreover, k-agreement can be interpreted as a constrained optimization problem with a non-separable cost function (see Section 2.2). In Pareto optimal distributed optimization [12], the group of agents cooperatively seeks to determine the minimizer of a cost function that depends on agent-dependent decision variables. Clustering-based consensus [1, 4, 23] is a closely related problem where the states of agents in the same cluster are related and states of agents in different clusters are independent. Instead, in k-agreement problems, the state of each agent is dependent on every other agent in the network. Scaled consensus [34], is a special case of k-agreement with $k = 1$. Interestingly, strong connectivity of the interaction graph is necessary and sufficient for scaled consensus; in contrast, in this paper, we show that strong connectivity is no longer sufficient when the dimension of the agreement space is $k \geq 2$.

Contributions. The contribution of this work is fourfold. *(c1)* We provide structural necessary conditions on the communication graph to reach a k-agreement on an arbitrary subspace; we apply this condition to study k-agreement protocols on basic graphs, such as path and circulant topologies. By drawing insights from our theorems, we show how these graphs can be modified to support k-agreement on high-dimensional subspaces. *(c2)* We provide a graph-theoretic sufficient condition to check if a group of agents can reach an arbitrary k-agreement; our analysis shows that k-agreement is made possible, graph-theoretically, by the presence of cycle families in the graph. *(c3)* We show how k-agreement algorithms can be adapted to account for cases where the local estimates are time-varying; this allows us to characterize the rate of convergence of k-agreement algorithms. *(c4)* We study the applicability of k-agreement protocols in robotic formation problems, and use these algorithms to constrain the asymptotic configuration of a team of robots.

Organization. Section 2 introduces the problem of interest. In Section 3, we present our main results: graph-theoretic conditions for k-agreement. Section 4 extends the framework to tracking problems, while Section 5 demonstrates the proposed methods through numerical simulations. Conclusions are provided in Section 6. For completeness, Appendices A–B review fundamental concepts from algebraic graph theory and linear subspaces that are used throughout the paper.

Notation. We let $\mathbb{N}_{>0} = \{1, 2, \dots\}$ denote the set of positive natural numbers. For $x \in \mathbb{C}$, $\Re(x)$ and $\Im(x)$ denote, respectively, its real and imaginary parts. When $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, $(x, u) \in \mathbb{R}^{n+m}$ denotes their concatenation. $\mathbb{1}_n \in \mathbb{R}^n$ is the vector of all ones; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; $\mathbb{0}_{n,m} \in \mathbb{R}^{n\times m}$ is the matrix of all zeros; subscripts may be dropped when dimensions are specified by the context. Given $A \in \mathbb{R}^{n\times n}$, we use the notation $A = [a_{ij}]$ to denote that a_{ij} is the element in row i and column j of A. For $A \in \mathbb{R}^{n\times n}$, $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$ denotes its spectrum, and $\lambda_{\max}(A) = \max\{\Re(\lambda) : \lambda \in \sigma(A)\}$ its spectral abscissa. When $A \in \mathbb{R}^{n\times m}$ is seen as a linear map, $\mathrm{Im}(A)$ denotes its image and $\ker(A)$ its null space. Given $p_1, \dots, p_n \in \mathbb{R}$, the polynomial $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + \dots + p_n$ is *stable* if all its roots have negative real part.

2 Problem setting

In this section, we formalize the problem of interest and motivate its applicability in multi-agent robotics. For completeness, we present basic notions on algebraic graph theory and linear projections used here in Appendices A-B.

2.1 Problem formulation

Consider a group of n cooperating agents that exchange information according to a directed communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each agent $i \in \{1, \dots, n\} = |\mathcal{V}|$ maintains a local variable $x_i \in \mathbb{R}$ and, at each time, updates it according to the following rule:

$$\dot{x}_i = a_{ii}x_i + \sum_{j\in\mathcal{N}_i} a_{ij}x_j, \tag{1}$$

where $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ denotes the set of agents that share their state variable with i. In (1), $a_{ij} \in \mathbb{R}$ are parameters describing the magnitude of the couplings between the agents. By setting $A = [a_{ij}]$, with $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, and $x = \mathrm{col}(x_1, \dots, x_n)$, (1) in vector form reads as:

$$\dot{x} = Ax. \tag{2}$$

Informally, we say that a network achieves a k-agreement if the agents' states converge to k independent weighted combinations of the initial conditions. This is made formal next.

Definition 2.1 *(k-agreement)* Let $n, k \in \mathbb{N}_{>0}$ with $k \leq n$, and let $W \in \mathbb{R}^{n\times n}$ satisfy $\mathrm{rank}(W) = k$. We say that the system (2) *globally asymptotically achieves a k-dimensional agreement on* W (or, for brevity, *achieves a k-agreement on* W) if, for any $x(0) \in \mathbb{R}^n$,

$$\lim_{t\to\infty} x(t) = Wx(0). \tag{3}$$

When (3) holds, A will be called a *k-agreement algorithm* (or *protocol*). □

Definition 2.1 formalizes a notion of agreement between the agents whereby, at convergence, the network's state is constrained to a k-dimensional linear subspace—precisely, the space $\mathrm{Im}(W)$.

Remark 2.2 ***(Link to Classical Consensus Frameworks)*** In the special case $k = 1$, the matrix W can be expressed as a rank-one matrix $W = vw^\top$ for some $v, w \in \mathbb{R}^n$. This corresponds to the well-known *scaled consensus* problem [34]. If, in addition, $v = \mathbb{1}$ and $w^\top\mathbb{1} = 1$, we recover

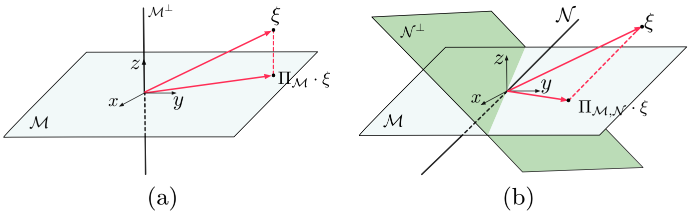

Fig. 1. (a) Geometric interpretation of orthogonal projections: a vector $\xi \in \mathbb{R}^3$ is projected onto $\mathcal{M} \subset \mathbb{R}^3$. (b) Geometric interpretation of oblique projections: $\mathcal{M}$ and $\mathcal{N} \in \mathbb{R}^3$ are complementary subspaces and ξ is projected on $\mathcal{M}$ along $\mathcal{N}$. Notice that the projection ray belongs to span$(\mathcal{N})$. Plot inspired by [24, Fig.1].

the standard *consensus* formulation—see [28]. In the particular case where $v = \mathbb{1}$ and $w = \frac{1}{n}\mathbb{1}$, the problem further simplifies to *average consensus* [28, Sec. 2]. We note that convergence of all state variables to a common value occurs only when $k = 1$ and $v = \mathbb{1}$. □

It was shown in [5, Prop. 4.2] that linear protocols of the form (2) can achieve a k-agreement only on matrices W that are oblique projections. See Fig. 1 for an illustration. Motivated by this result, we introduce the following assumption.

Assumption 1 ***(Matrix of weights is a projection)*** *The matrix W in* (3) *satisfies $W^2 = W$ and* $\operatorname{rank}(W) = k$. □

Under Assumption 1, there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that (see Lemma B.1):

$$W = T \begin{bmatrix} I_k & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} T^{-1}. \tag{4}$$

In what follows, we denote by $t_1, \dots, t_n$ the columns of T and by $\tau_1^\mathsf{T}, \dots, \tau_n^\mathsf{T}$ the rows of T^{-1}. Consequently to Assumption 1, k-agreement protocols find applications in problems where the goal is to compute an oblique projection of the network's initial conditions. See Section 2.2 and Remark 2.4 for a discussion of illustrative applications.

In this work, we aim to address the following question: given an arbitrary matrix of weights W, which communication topologies $\mathcal{G}$ allow the agents to achieve a k-agreement on W? This question motivates the following definition.

Definition 2.3 ***(k-agreement reachability)*** Let $k \in \mathbb{N}_{>0}$ and consider a set of n agents with communication graph $\mathcal{G}$. The set of agents is said to be:

- *k-agreement reachable on some weights* if there exists a matrix $W \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(W) = k$, such that there exists a matrix $A \in \mathbb{R}^{n\times n}$, consistent with $\mathcal{G}$, for which the system (2) achieves a k-agreement on W.
- *k-agreement reachable on arbitrary*[1] *weights* if, for any matrix $W \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(W) = k$, there exists a matrix $A \in \mathbb{R}^{n\times n}$, consistent with $\mathcal{G}$, such that (2) achieves a k-agreement on W. □

With these definitions in place, the question posed above can now be formalized as follows.

Problem 1 *Given $k \in \mathbb{N}_{>0}$ and a group of agents communicating over a graph $\mathcal{G}$, develop a method to determine whether the group is k-agreement reachable on arbitrary weights.* □

We note that for directed graphs and $k = 1$, Problem 1 has been solved: a group of agents is 1-agreement reachable on arbitrary weights if and only if the graph is strongly connected [27, 34]. In contrast, identifying which topologies enable the agents to reach a k-agreement on arbitrary weights remains an open problem.

A necessary condition for k-agreement reachability on arbitrary weights is that the underlying communication graph $\mathcal{G}$ is strongly connected (see [5, Lem.4.5]). Intuitively, since W is arbitrary, each component of the limiting state vector $x(t)$ as $t \to \infty$ generally depends on the entire initial state vector $x(0)$ (cf. (3)). Therefore, information from every agent must be able to reach the entire network—thus requiring strong connectivity. We thus make the following assumption.

Assumption 2 ***(Strong connectivity)*** *The communication digraph $\mathcal{G}$ is strongly connected. Moreover, each node in $\mathcal{G}$ has a self cycle*[2] *.* □

It is important to note that strong connectivity of the communication graph alone does not guarantee k-agreement reachability on arbitrary weights. In fact, it was shown in [5, Example 4.6] that for a network of three agents aiming for a 2-agreement, agreement on arbitrary weights is only possible if the graph is complete. It follows that, in sparse graphs, structural limitations on the protocol A can prevent the achievement of k-agreement.

2.2 Illustrative application: formation control

To illustrate the importance of designing k-agreement algorithms, we next demonstrate how this problem provides a natural solution to enforce a desired configuration in multi-agent mobile robotics. Consider a group of $n = 4$ robots modeled using single-integrator dynamics. Let $x(0) \in \mathbb{R}^4$ denote the **x**-coordinates of the robots' positions at time 0 (we refer to Section 5 for a generalization accounting also for the **y**-coordinates), and assume that the group is interested in achieving a final formation $x^* = \operatorname{col}(x_1^*, x_2^*, x_3^*, x_4^*)$ such that $x_1^* = x_2^*, x_3^* = x_4^*$, while minimizing the distance from the initial state of the robots. Formally, the desired configuration is the solution of the optimization problem:

$$\begin{aligned} x^* = \arg\min_{x\in\mathbb{R}^4} \quad & \|x(0) - x\|_R^2 \\ \text{subject to:} \quad & x_1 = x_2, \ x_3 = x_4, \end{aligned} \tag{5}$$

where $\|\cdot\|_R^2$ denotes the square weighted norm defined by $\mathbb{R}^{4\times 4} \ni R \succ 0$. Further, because each robot has no knowledge of global coordinates, this must be achieved through a distributed coordination algorithm. By letting

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

the formation requirements can be encapsulated by the constraint $Dx = 0$, and the solution to (5) with $R = I$ is given by $x^* = \Pi_{\ker(D)} x(0)$. It is now immediate to see that x^* can be computed using a k-agreement algorithm, with

[1] We remark that, in this definition, the term "arbitrary" refers to the matrix W, not to the matrix A.

[2] We observe that incorporating self-cycles is a standard practice in coordination protocols of the form (2), since leveraging a node's own state in its update incurs no additional communication cost.

$W = \Pi_{\ker(D)}$. We note that although formation control problems have also been addressed using consensus algorithms combined with relative positioning (e.g., [31]) or by incorporating additional inputs to track global variables (e.g., [10]), the use of k-agreement protocols presents several advantages. First, k-agreement protocols guarantee optimality as specified by (5). Second, they significantly reduce communication complexity compared to approaches that require estimation of multiple global quantities (see [5, Sec.III.B] for a comparison of communication complexities). Third, k-agreement protocols provide greater generality by implicitly specifying the desired configuration through the mathematical optimization problem (5), thereby eliminating the need for explicit relative positioning.

We conclude this section by discussing engineering applications of projections and extensions in the following remarks.

Remark 2.4 ***(Relevance of Projections in Engineering Applications)*** The computation of linear orthogonal and oblique projections is a problem that arises naturally in a range of engineering applications [2]. Orthogonal projections, for instance, are widely used in standard regression problems (cf. Section 6), whereas oblique projections are essential in weighted or constrained least-squares formulations [2, 11]. These projections also play a key role in areas such as signal processing [3], subspace system identification [14], and other related domains [2]. □

Remark 2.5 ***(Extensions to projections onto affine subspaces)*** Assume that the agents aim to achieve $\lim_{t\to\infty} x(t) = Wx(0)+v$, where $v = \mathrm{col}(v_1, \dots, v_n) \in \mathbb{R}^n$ is a given translation vector. Starting from an algorithm of the form (1) that computes a k-agreement, one can construct a modified algorithm that projects onto the desired affine subspace by introducing an additional local variable y_i, defined as $y_i = x_i + v_i$. This coupling effectively shifts the original agreement dynamics to target the translated subspace. □

3 Structural conditions for k-agreement

In this section, we provide necessary and a sufficient condition for k-agreement reachability on arbitrary weights. We begin with the following necessary condition.

Theorem 3.1 ***(Graph-theoretic necessary conditions)*** *Let W be a matrix satisfying Assumption 1, and $\mathcal{G}$ be a graph satisfying Assumption 2. The set of agents is globally k-agreement reachable on arbitrary weights only if*

$$|\mathcal{E}| \geq kn. \tag{6}$$

□

PROOF. Let $t_1, \dots, t_n$ and $\tau_1, \dots, \tau_n$ be as in (4), and A be a matrix consistent with $\mathcal{G}$. By Theorem A.2, $\dot{x} = Ax$ reaches a k-agreement if and only if A satisfies:

$$0 = At_i, \quad \tau_i^{\mathsf{T}} A = 0, \qquad\qquad i \in \{1, \dots, k\}, \tag{7a}$$

$$p_\ell = \sum_{\xi \in \mathcal{C}_\ell(\mathcal{G})} (-1)^{d(\xi)} \prod_{(i,j)\in\xi} a_{ij}, \quad \ell \in \{1, \dots, n-k\}, \tag{7b}$$

where $d(\xi)$ denotes the number of cycles in the cycle family ξ, and $p_1, \dots, p_{n-k}$ are real numbers that can be freely chosen, provided that the polynomial $\lambda^{n-k} + p_1\lambda^{n-k-1} + \dots + p_{n-k}$ is stable. The first set of equations (7a) consists of nk independent scalar linear equations and $|\mathcal{E}|$ unknowns. Equations (7a) are linearly independent and thus the generic solvability of (7) requires the following necessary condition: $|\mathcal{E}| \geq nk$, from which the claim follows. ■

The inequality (6) provides two important types of insights. First, given n and k, (6) gives a lower bound on the minimal graph connectivity (i.e., $|\mathcal{E}|$) required for k-agreement. As k increases, (6) states that the number of edges in $\mathcal{G}$ must grow at least linearly with k. Second, given a network topology $\mathcal{G}$ (i.e., given n and $\mathcal{E}$), (6) gives an upper bound on the dimension of the allowable agreement space: $k \leq |\mathcal{E}|/n$. These bounds provide valuable insights into the interplay between agreement spaces and graph topologies, as demonstrated in the following examples.

Example 3.2 ***(k-agreement reachability in circulant networks)*** Consider the one-directional circulant topology of Fig. 2(a) (note that self-loops are omitted in the plots of Fig. 2, for illustration purposes). In this case, $|\mathcal{E}| = 2n$ and thus (6) gives $k \leq 2$. Next, consider the bi-directional circulant topology of Fig. 2(b). Here, $|\mathcal{E}| = 3n$, and (6) gives $k \leq 3$. In other words, these bounds state that: agents interacting through a one-directional circulant digraph can compute projections on arbitrary linear subspaces of dimension at most 2; similarly, agents interacting through a bi-directional circulant digraph can compute projections on arbitrary linear subspaces of dimension at most 3.

Generalizing this idea, consider a bi-directional circulant digraph where each agent communicates with $\alpha \in \mathbb{N}_{>0}$ nearest neighbors (see Fig. 2(c)). Using (6) with $|\mathcal{E}| = n(\alpha + 1)$, gives $\alpha \geq k - 1$. In other words, in circulant-type communication topologies, to compute projections on arbitrary linear subspaces of dimension k, each agent needs to communicate with at least $k - 1$ independent neighbors. □

Example 3.3 ***(k-agreement reachability in path networks)*** Consider the bi-directional path topology of Fig. 2(d). In this case, $|\mathcal{E}| = n + 2(n - 1)$ and (6) yields $k \leq \lfloor \frac{3n-2}{n} \rfloor \leq 3$. In analogy with the bi-directional circulant topology, agents interacting through a bi-directional path topology can compute projections on arbitrary linear subspaces of dimension at most 3. Generalizing, consider bi-directional path digraphs where each agent communicates with $\alpha \in \mathbb{N}_{>0}$ nearest neighbors (see Fig. 2(e)). Using (6) with $|\mathcal{E}| = n + \alpha n - \frac{\alpha}{2}(\frac{\alpha}{2} + 1)$, gives $\alpha \geq 2k - 1$.

By comparison with Example 3.2, k-agreement protocols on path topologies require higher connectivity than those on circulant digraphs. Specifically, the former necessitate $\alpha \geq 2k - 1$, whereas the latter require only $\alpha \geq k - 1$. □

We discuss the relationship between (6) and established conditions for consensus protocols in the following remark.

Remark 3.4 ***(Strong connectivity implies*** (6) ***when*** $k = 1$***)*** Recall that a group of agents is 1-agreement reachable on arbitrary weights if and only if the communication digraph is strongly connected (see the discussion following Problem 1). We now show that strong connectivity implies condition (6) in this case. Note that the strongly connected digraph with the minimal number of edges (including self-loops) is the one-directional circulant graph shown in Fig.2(a), which has $|\mathcal{E}| = 2n$. Applying (6) yields

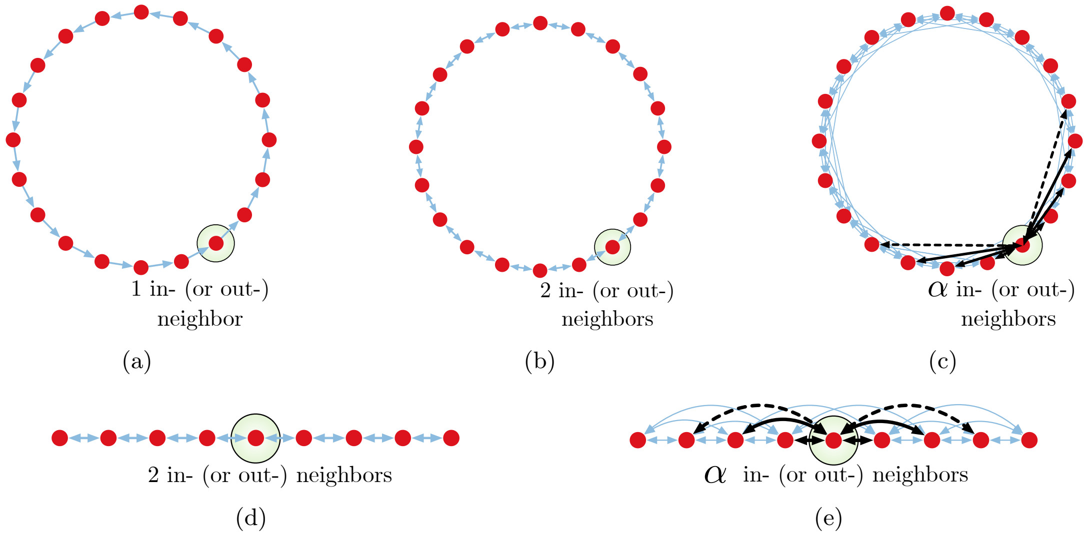

Fig. 2. (a) One-directional circulant topology; (b)–(c) bi-directional circulant topology; (d)-(e) bi-directional path topology. The graph in (a) is the least-connected graph topology that can reach a 1-agreement on arbitrary weights (see Remark 3.4). (b) and (d) also admit k-agreement protocols on arbitrary weights within subspaces of dimension at most $k = 1$ (see Examples 3.2, 3.3). For (c) and (d), the connectivity of each node α must scale proportionally with k. (see examples 3.2 and 3.3). In all plots, all nodes have self-cycles, which are omitted here for illustration purposes. Dashed lines illustrate the trend of edge increase as a function of α.

$2n \geq 1 \cdot n$, which holds for any n. Thus, the strong connectivity condition—necessary and sufficient for 1-agreement reachability on arbitrary weights—automatically ensures that the necessary condition (6) is met. □

The following result provides structural sufficient-conditions for k-agreement reachability on arbitrary weights. To state the result, we recall that the notation $\mathcal{C}_\ell(\mathcal{G})$ denotes the set of all cycle families of length ℓ in $\mathcal{G}$ (see Appendix A).

Theorem 3.5 ***(Graph-theoretic sufficient conditions)*** *Let W be a matrix satisfying Assumption 1, and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph satisfying Assumption 2. Suppose $|\mathcal{E}| \geq nk+n-k$. Suppose that there exists a partitioning of the edge set $\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_c$, $\mathcal{E}_v \cap \mathcal{E}_c = \emptyset$, with $|\mathcal{E}_v| = n-k$ and $|\mathcal{E}_c| = |\mathcal{E}| - n + k$, such that, for all $\ell \in \{1, \dots, |\mathcal{E}_c|\}$, there exists $\mathcal{C}^*_\ell \in \mathcal{C}_\ell(\mathcal{G})$ that satisfies:*

(i) $|\mathcal{C}^*_\ell \cap \mathcal{E}_v| = 1$;
(ii) $\mathcal{C}^*_\ell \setminus (\mathcal{E}_v \cap \mathcal{C}^*_\ell) \subseteq \mathcal{E}_c$;
(iii) $\mathcal{C}^*_\ell \cap \mathcal{E}_v \cap \mathcal{C}^*_j = \emptyset, j > \ell$.

Then, the set of agents is globally k-agreement reachable on arbitrary weights. □

PROOF. Given an arbitrary matrix W, and under assumptions *(i)–(iii)*, we present a construction of a matrix A such that the system (2) achieves a k-agreement on W. Let $t_1, \dots, t_n$ and $\tau_1, \dots, \tau_n$ be the vectors defined in (4), and A be a matrix consistent with $\mathcal{G}$. By Theorem A.2, $\dot{x} = Ax$ reaches k-agreement if and only if A satisfies:

$$0 = At_i, \quad \tau_i^\mathsf{T} A = 0, \qquad i \in \{1, \dots, k\}, \tag{8a}$$
$$p_\ell = \sum_{\xi \in \mathcal{C}_\ell(\mathcal{G})} (-1)^{d(\xi)} \prod_{(i,j)\in\xi} a_{ij}, \quad \ell \in \{1, \dots, n-k\}, \tag{8b}$$

where $d(\xi)$ denotes the number of cycles in the cycle family ξ, and $p_1, \dots, p_{n-k}$ are real numbers that can be freely chosen, provided that the polynomial $\lambda^{n-k} + p_1\lambda^{n-k-1} + \dots + p_{n-k}$ is stable. By the Routh–Hurwitz stability criterion, all the coefficients $\{p_1, \dots, p_{n-k}\}$ are required to be strictly positive; hence, in what follows, we interpret $\{p_1, \dots, p_{n-k}\}$ as free variables and harness the Inverse Function Theorem [35, Thm. 9.24] to show that there always exist a choice of A that satisfies the set of equations (8) everywhere in a neighborhood of $p_1 = p_2 = \dots = p_{n-k} = 0$.

With a slight abuse of notation, we let $e_c = (e_{c,1}, e_{c,2}, \dots)$ [resp. $e_v = (e_{v,1}, e_{v,2}, \dots)$] be a vector such that each component $e_{c,i}$ [resp. $e_{v,i}$] corresponds to a real-valued quantity associated with the i-th element of $\mathcal{E}_c$ [resp. $\mathcal{E}_v$]. Equation (8a) defines a set of nk linearly independent equations in the variables (e_c, e_v), which we denote in compact form by $0 = h(e_c, e_v)$, where $h : \mathbb{R}^{|\mathcal{E}_c|} \times \mathbb{R}^{|\mathcal{E}_v|} \to \mathbb{R}^{nk}$. Let $p := (p_1, \dots, p_{n-k})$. The set of equations (8b) relates the vector p with the elements of $\mathcal{E}_c \cup \mathcal{E}_v$ by means of a nonlinear mapping $p = g(e_c, e_v)$, where $g : \mathbb{R}^{|\mathcal{E}_c|} \times \mathbb{R}^{|\mathcal{E}_v|} \to \mathcal{T}$ is a smooth mapping and $\mathcal{T}$ is smooth manifold in $\mathbb{R}^{n-k}$. Since $g(\cdot)$ is a multi-linear polynomial in the variables e_c, e_v, it is immediate to verify that the following properties hold:

$$\frac{\partial g(e_c, e_v)}{\partial e_v} \cdot e_v = g(e_c, e_v). \tag{9}$$

Denote the Jacobian matrices of h and g by:

$$H(e_c, e_v) := \frac{\partial h(e_c, e_v)}{\partial e_v} \in \mathbb{R}^{nk \times n-k},$$
$$G(e_c, e_v) := \frac{\partial g(e_c, e_v)}{\partial e_v} \in \mathbb{R}^{n-k \times n-k}.$$

Then, using (9) and the linearity of (8a), the system of equations (8) can equivalently be rewritten as:

$$0 = H(e_c, e_v)e_v, \qquad p = G(e_c, e_v)e_v. \tag{10}$$

According to the Inverse Function Theorem [35, Thm. 9.24], the local solvability of (10) in a neighborhood of $p = 0$ is ensured provided there exist $e_c^* \in \mathrm{R}^{|\mathcal{E}_c|}$ and $e_v^* \in \mathrm{R}^{|\mathcal{E}_v|}$ such that (a) $0 = H(e_c^*, e_v^*)e_v^*$, (b) $0 = G(e_c^*, e_v^*)e_v^*$, and (c) the matrix $G(e_c^*, e_v^*)$ is invertible. To establish these three properties, note that, by (9), for any $e_c \in \mathrm{R}^{|\mathcal{E}|-n+k}$, we have $0 = H(e_c^*, e_v^*)e_v^*$ and $0 = G(e_c^*, e_v^*)e_v^*$; in other words, conditions (a) and (b) hold with the choice $e_v^* = 0$ and e_c^* arbitrary Next, we will establish (c) by constructing, via induction, a vector e_c^* for which the matrix $G(e_c^*, e_v^*)$ is invertible; more precisely, we will construct e_c^* such that $G(e_c^*, e_v^*)$ is a lower triangular matrix with a nonzero diagonal.

(Base case.) Let the elements of $\mathcal{E}_v$ be $\mathcal{E}_v = \{\mathcal{E}_{v,1}, \mathcal{E}_{v,2}, \dots\}$, and suppose that they are ordered such that $\mathcal{E}_{v,\ell} \in \mathcal{C}_\ell^*$. Consider the matrix $G(e_c, e_v^*)$, and partition it as follows:

$$G(e_c, e_v^*) = \begin{bmatrix} G_{11}(e_c, e_v^*) & G_{12}(e_c, e_v^*) \\ G_{21}(e_c, e_v^*) & G_{22}(e_c, e_v^*) \end{bmatrix},$$

with $G_{11}(e_c, e_v^*) \in \mathbb{R}, G_{12}(e_c, e_v^*) \in \mathbb{R}^{1\times(n-k-1)}, G_{21}(e_c, e_v^*) \in \mathbb{R}^{(n-k-1)\times 1}$, and $G_{22}(e_c, e_v^*) \in \mathbb{R}^{(n-k-1)\times(n-k-1)}$. By (8b) with $\ell = 1$ and condition *(ii)* of the statement, it follows that $G_{11}(e_c, e_v^*) = -1$. Moreover, by condition *(iii)* of the statement, we obtain $G_{12}(e_c, e_v^*) = 0$. We have thus shown that, if $G_{22}(e_c, e_v^*)$ is invertible, then $G(e_c, e_v^*)$ is also invertible.

(Induction step.) Let $i \in \{1, \dots, n-k-1\}$ and $G^{(i)}(e_c, e_v^*)$ denote the submatrix of dimensions $i \times i$ located in the bottom-right corner of $G(e_c, e_v^*)$. Partition this matrix as:

$$G^{(i)}(e_c, e_v^*) = \begin{bmatrix} G_{11}^{(i)}(e_c, e_v^*) & G_{12}^{(i)}(e_c, e_v^*) \\ G_{21}^{(i)}(e_c, e_v^*) & G_{22}^{(i)}(e_c, e_v^*) \end{bmatrix},$$

where $G_{11}^{(i)}(e_c) \in \mathbb{R}$, $G_{12}^{(i)}(e_c) \in \mathbb{R}^{1\times(i-1)}$, $G_{21}^{(i)}(e_c) \in \mathbb{R}^{(i-1)\times 1}$, and $G_{22}^{(i)}(e_c) \in \mathbb{R}^{(i-1)\times(i-1)}$. By (8b) and *(i)*, $G_{11}^{(i)}(e_c)$ is given by a single term: the product of edge weights along the cycle family $\mathcal{C}_\ell^*, \ell = n-k-i+1$ (excluding the edge weight of $\mathcal{E}_{v,\ell}$). By condition *(ii)* of the statement, this product depends only on edge weights in $\mathcal{E}_c$; hence, by letting e_c^* be any vector such that all its entries are nonzero, we conclude that $G_{11}^{(i)}(e_c^*, e_v^*) \neq 0$. Moreover, by *(iii)*, $G_{12}^{(i)}(e_c^*, e_v^*) = 0$. We have thus shown that, if $G_{22}^{(i)}(e_c^*, e_v^*)$ is invertible, then $G^{(i)}(e_c^*, e_v^*)$ is also invertible. The claim follows by iterating the reasoning for all $i \in \{1, \dots, n-k-1\}$, which completes the proof. ■

Theorem 3.5 establishes a set of conditions that ensure a set of agents is k-agreement reachable. These conditions are *sufficient*, but not *necessary.* In particular, the requirement $|\mathcal{E}| \geq nk + n - k$ is more conservative than the necessary bound given in (6). In words, the conditions *(i)-(iii)* in the theorem statement can be interpreted as follows:

• Condition *(i)* states that, for all $\ell \in \{1, \dots, n-k\}$, there exists a cycle family of length ℓ with exactly one edge in $\mathcal{E}_v$;

• By condition *(ii)*, every edge in $\mathcal{C}_\ell^*$ that is not in the intersection $\mathcal{E}_v \cap \mathcal{C}_\ell^*$ is necessarily contained in $\mathcal{E}_c$; and

• Condition *(iii)* states that the edge in $\mathcal{C}_\ell^*$ that is in $\mathcal{E}_v$ does not appear in any other cycle family $\mathcal{C}_j^*$ with $j > \ell$.

The applicability of Theorem 3.5 hinges on the ability to partition the edge set $\mathcal{E}$ into two subsets, $\mathcal{E}_v$ and $\mathcal{E}_c$. An

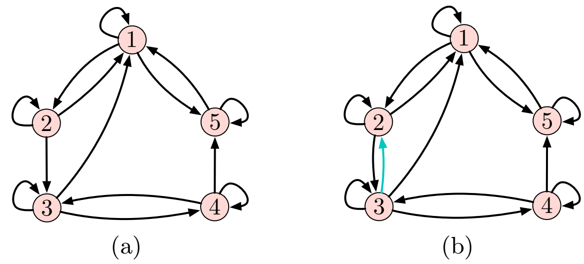

Fig. 3. (a) Example of a graph that admits a 2-dimensional agreement protocol on arbitrary weights. (b) Graph obtained by adding green edges to (a); this graph admits a 3-agreement protocol on arbitrary weights. See Example 3.7.

algorithm to verify the existence of such a partition can be developed by leveraging techniques similar to those in [37], wherein $\mathcal{E}_v$ and $\mathcal{E}_c$ are derived from a directed spanning tree of the graph $\mathcal{G}$. Due to space constraints, we refer to [37] (and pertinent references therein) for the detailed algorithm.

We conclude this section by discussing how Theorem 3.5 modifies under edge addition, and by demonstrating its applicability through an example.

Remark 3.6 ***(k-agreement reachability under edge addition)*** It is important noting that cycle families do not vanish under edge addition; thus, if *(i)-(iii)* hold for a certain graph $\mathcal{G}$, they continue to hold for any other graph obtained by edge addition. To see this, denote by $\mathcal{C}_\ell(\mathcal{G})$ the set of cycle families of length ℓ of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose that $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is any graph such that $\mathcal{V}' = \mathcal{V}$ and $\mathcal{E} \subset \mathcal{E}'$. Since no edge has been removed, the set of cycle families of length ℓ of $\mathcal{G}'$ satisfies $\mathcal{C}_\ell(\mathcal{G}) \subseteq \mathcal{C}'_\ell(\mathcal{G})$. It follows that, if the agents are k-agreement reachable on arbitrary weights when interacting through $\mathcal{G}$, then they are also k-agreement reachable on arbitrary weights when the interaction graph is any graph obtained by adding edges to $\mathcal{G}$. □

Example 3.7 ***(Illustration of the conditions in Theorem 3.5)*** Consider the communication digraph of Fig. 3(a). The set of all k-agreement protocols compatible with this graph is given by the parametric matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

By Theorem 3.1, a necessary condition for k-agreement is

$$k \leq \left\lfloor \frac{|\mathcal{E}|}{n} \right\rfloor = \left\lfloor \frac{14}{5} \right\rfloor = 2.$$

Hence, we will select $k = 2$. To illustrate the conditions of Theorem 3.5, for simplicity, we let $a_{22} = a_{33} = a_{44} = a_{55} = 0$ (according to Remark 3.6, if the graph without self-cycles has an independent set of cycle families, then the graph obtained by adding these self-cycles will retain the same set of decompositions). With this choice, the set of cycle families of length $\ell \in \{1, \ldots, 3\}$, $\ell \in \{1, \ldots, n-k\}$, is:

$$
\begin{aligned}
&\mathcal{C}_1(\mathcal{G}) = \{\{a_{11}\}\},\\
&\mathcal{C}_2(\mathcal{G}) = \{\{a_{12}, a_{21}\}, \{a_{34}, a_{43}\}, \{a_{15}, a_{51}\}\},\\
&\mathcal{C}_3(\mathcal{G}) = \{\{a_{11}, a_{34}, a_{43}\}, \{a_{13}, a_{21}, a_{32}\}\}.
\end{aligned} \tag{11}
$$

By letting $\mathcal{E}_v$ and $\mathcal{E}_c$:

$$
\begin{aligned}
&\mathcal{E}_v = \{a_{11}, a_{12}, a_{13}\},\\
&\mathcal{E}_c = \{a_{51}, a_{54}, a_{21}, a_{32}, a_{34}, a_{43}, a_{15}\},
\end{aligned}
$$

it is immediate to verify that a set of cycle families $\mathcal{C}_1^*, \mathcal{C}_2^*, \mathcal{C}_3^*$ that satisfies conditions *(i)-(iii)* of Theorem 3.5 is given by:

$$
\mathcal{C}_1^* = \{a_{11}\}, \quad \mathcal{C}_2^* = \{a_{12}, a_{21}\}, \quad \mathcal{C}_3^* = \{a_{13}, a_{21}, a_{32}\}.
$$

Indeed, with this choice, the set of equations (7b) reads as:

$$
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -a_{21} & 0 \\ a_{34}a_{43} & 0 & -a_{21}a_{32} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} - \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix},
$$

where $\gamma = a_{34}a_{43} + a_{15}a_{51}$, which is generically solvable for any $(p_1, p_2, p_3) \in \mathbb{R}^3$. Any choice of weights such that $a_{21} \neq 0$ and $a_{21}a_{32} \neq 0$ guarantees that the above matrix is invertible and thus that the set of equations is solvable.

To achieve k-agreements on subspaces of dimension $k = 3$, consider the graph in Fig. 3(b), obtained by adding edges to the graph of Fig. 3(a). The necessary condition (6) yields

$$
k \leq \left\lfloor \frac{|\mathcal{E}|}{n} \right\rfloor = \left\lfloor \frac{15}{5} \right\rfloor = 3,
$$

which is satisfied. The set of cycle families (11) shall be modified to:

$$
\begin{aligned}
&\mathcal{C}_1 = \{\{a_{11}\}\},\\
&\mathcal{C}_2 = \{\{a_{12}, a_{21}\}, \{a_{34}, a_{43}\}, \{a_{15}, a_{51}\}, \{a_{23}, a_{32}\}\}.
\end{aligned}
$$

By selecting $\mathcal{E}_v$ and $\mathcal{E}_c$ as follows

$$
\begin{aligned}
&\mathcal{E}_v = \{a_{11}, a_{12}\},\\
&\mathcal{E}_c = \{a_{13}, a_{23}, a_{45}, a_{35}, a_{51}, a_{54}, a_{21}, a_{32}, a_{34}, a_{43}, a_{15}\},
\end{aligned}
$$

a set of cycle families that satisfies Theorem 3.5 is:

$$
\mathcal{C}_1^* = \{a_{11}\}, \qquad \mathcal{C}_2^* = \{a_{12}, a_{21}\},
$$

thus showing that the sufficient conditions also hold. □

4 Convergence rates and extensions to tracking dynamics for k-agreement

Analogous to dynamic consensus protocols [22], k-agreement algorithms can be modified to track the oblique projection of a time-varying forcing signal $u(t)$ (in place of $x(0)$ as in (3)). By studying these extensions, we also obtain explicit convergence rates for k-agreement algorithms. The study of k-agreement tracking problems is motivated by practical applications such as robotic formation control (among others, see [22]), where a team of agents seeks to follow a moving target while preserving a specific formation represented as a point constrained to lie within a k-dimensional subspace.

Given a digraph $\mathcal{G}$, consider the network process:

$$
\dot{x} = Ax + \dot{u}, \qquad x(0) = u(0), \tag{12}
$$

where A is chosen so that (3) holds and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a continuously-differentiable function. In this framework, the i-th entry of $\dot{u}$ is known only by the agent i, and the goal is to design an algorithm with state $x(t)$ that tracks $Wu(t)$, asymptotically. The protocol (12) can be interpreted as a generalization of the dynamic average consensus algorithm [22, eq. (11)], where the communication matrix is an agreement matrix instead than a Laplacian. The following result characterizes the transient behavior of (12).

Proposition 4.1 ***(Convergence of dynamic k-agreement protocols)*** *Consider equation* (12) *and let A be such that* (3) *holds. Then, for all $t \geq 0$:*

$$
\|x(t) - Wu(t)\| \leq e^{-\hat{\lambda}t} \|x(0) - Wu(0)\| + \frac{1}{\hat{\lambda}} \sup_{0 \leq \tau \leq t} \|\dot{u}(\tau)\|, \tag{13}
$$

where $\hat{\lambda} := \lambda_{\max}\left(\frac{A + A^\mathsf{T}}{2}\right)$. □

PROOF. The proof is inspired from [22, Thm. 2] and extends the result to non Laplacian-based protocols and non weight-balanced digraphs. Let W be decomposed as in (4), and consider the following decompositions for T and T^{-1}:

$$
T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \qquad (T^{-1})^\mathsf{T} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \tag{14}
$$

where $T_1, U_1 \in \mathbb{R}^{n \times k}$ and $T_2, U_2 \in \mathbb{R}^{n \times n-k}$. Let $e = x - Wu$ denote the tracking error, and consider the change of variables $\bar{e} = T^{-1}e$. In the new variables:

$$
\begin{aligned}
\dot{\bar{e}} &= T^{-1}(\dot{x} - W\dot{u})\\
&= T^{-1}AT\bar{e} + T^{-1}AWu + T^{-1}\dot{u} - T^{-1}W\dot{u},\\
&= T^{-1}AT\bar{e} + T^{-1}\dot{u} - T^{-1}W\dot{u},
\end{aligned}
$$

where the last identity follows from (4), which implies $AW = 0$. By substituting (14) and by noting that $T^{-1}W = [U_1 \ 0]^\mathsf{T}$:

$$
\begin{aligned}
\dot{\bar{e}} &= \begin{bmatrix} U_1^\mathsf{T}AT_1 & U_1^\mathsf{T}AT_2 \\ U_2^\mathsf{T}AT_1 & U_2^\mathsf{T}AT_2 \end{bmatrix} \bar{e} + \begin{bmatrix} 0 \\ U_2^\mathsf{T} \end{bmatrix} \dot{u}\\
&= \begin{bmatrix} 0 \\ U_2^\mathsf{T}AT_2 \end{bmatrix} \bar{e} + \begin{bmatrix} 0 \\ U_2^\mathsf{T} \end{bmatrix} \dot{u},
\end{aligned} \tag{15}
$$

where the last inequality follows by noting that $0 = U_1^\mathsf{T}AT_1 = U_1^\mathsf{T} = AT_1$ according to [5, Thm. 5.3 - cond. (i)].

Next, decompose $e = (e_1, e_2)$ and $\bar{e} = (\bar{e}_1, \bar{e}_2)$, where $e_1, \bar{e}_1 \in \mathbb{R}^k$ and $e_2, \bar{e}_2 \in \mathbb{R}^{n-k}$, and notice that the following identities hold:

$$
\bar{e}_2 = U_2^\mathsf{T} e, \qquad e = T_2 \bar{e}_2. \tag{16}
$$

The first identity follows immediately from (14), while the second follows from (14) and $\bar{e}_1(t) = 0$ at all times. To see that $\bar{e}_1(t) = 0\ \forall t \geq 0$, notice that $\bar{e}_1(0) = U_1^\mathsf{T}(x(0) - u(0)) = 0$ thanks to the initialization (12), and $\dot{\bar{e}}_1 = 0$ according to (15). By using (16), we conclude that $\dot{e} = Ae + \dot{u}$, from which (13) follows by noting that

$$e(t) = \exp(At) \cdot e(0) + \int_0^t \exp(A(t-\tau))B\dot{u}(\tau)d\tau,$$

and using $\|\exp(At)\| \leq \exp\left(-\lambda_{\max}\left(\frac{A+A^\mathsf{T}}{2}\right)t\right)$. ■

In other words, the error bound (13) states that the dynamics (12) are input-to-state stable [38] with respect to $\dot{u}$. It follows that, for any $u(t)$ with bounded time-derivative, the tracking error $\|x(t) - Wu(t)\|$ is bounded at all times. As a special case, if $\lim_{t\to\infty} u(t) = u^* \in \mathbb{R}^n$, then $\lim_{t\to\infty} x(t) = Wu^*$ (since $\lim_{t\to\infty} \dot{u}(t) = 0$). The bound (13) also provides an estimate of the convergence rate (in both the dynamic and static setting, the latter as a special case of the former), showing that it is, as measured by the 2-norm, governed by the spectral properties of the matrix $A + A^\mathsf{T}$.

5 Applications to robotic formation control

We next illustrate the applicability of k-agreement protocols to solve formation problems [26] in multi-robot systems inspired from [24]. Consider a team of $n = 8$ planar single-integrator robots initially arranged on a unit circle (illustrated by the gray lines in Fig. 4(a)-(c)). By using **x**- and **y**-coordinates to describe the robots' positions, the network's initial state is given by: $x_0 = (\cos(0), \sin(0), \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}), \ldots, \cos(\frac{7\pi}{4}), \sin(\frac{7\pi}{4})) \in \mathbb{R}^{16}$. To account for planar coordinates, the state of (2) is partitioned into **x** and **y** coordinates, and the algorithm (2) is:

$$\dot{x} = (A \otimes I_2)x, \qquad x(0) = x_0.$$

For our simulations, we utilized the circulant communication topology in Fig. 2(c) with $\alpha = 4$, and k-agreement protocols A have been constructed by solving numerically the set of equations (8b). Simulation results are shown in Fig. 4.

In Fig. 4(a) and (d), we report the state trajectories obtained by choosing $k = 1$ and $W = \frac{1}{n}\mathbb{1}\mathbb{1}^\mathsf{T}$. Notice that, in this special case, the k-agreement algorithm simplifies to an average consensus algorithm [28]; as expected for consensus, the robots meet at $(0, 0)$, which coincides with the average of the initial conditions. This special case corresponds to the robotic rendezvous problem [26]. In Fig.4(b) and (e), we illustrate the state trajectories obtained by choosing $k = 3$ and $W = \Pi_\mathcal{M}$, where $\Pi_\mathcal{M}$ is the orthogonal projection onto $\mathcal{M} = \ker(M_1)$, with

$$M_1 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}.$$

The matrix M_1 encodes attraction and repulsion forces between the robots at convergence. Indeed, from $x(\infty) \in \ker(M_1 \otimes I_2)$, it follows that:

$$\begin{aligned} \mathbf{x}_1(\infty) + \mathbf{x}_4(\infty) &= \mathbf{x}_2(\infty) + \mathbf{x}_3(\infty), \\ \mathbf{y}_1(\infty) + \mathbf{y}_4(\infty) &= \mathbf{y}_2(\infty) + \mathbf{y}_3(\infty). \end{aligned} \tag{17}$$

Simulation results are illustrated in Fig.s 4(b) and (e). Finally, we illustrate in Fig.s 4(d) and (f) the robots' trajectories obtained by choosing the oblique projection: $W = \Pi_{\mathcal{M},\mathcal{N}}$, $\mathcal{M} = \ker(M_1)$, $\mathcal{N} = \operatorname{Im}(N_1)$, where

$$N_1^\mathsf{T} = \begin{bmatrix} -1 & 5 & 5 & -1 \end{bmatrix}.$$

The use of an oblique projection can be interpreted as a non-homogeneous weighting for the vector that defines the final configuration. As depicted in the figure, the rendezvous is no longer midway for all robots: robots 2 and 3 (respectively, 6 and 7) cover a larger distance than robots 1 and 4 (respectively, 5 and 8)

6 Conclusions

We investigated the conditions under which a group of agents can achieve a k-agreement with arbitrary weighting. We showed that k-agreement protocols require a high level of network connectivity, which must scale with the dimension k of the agreement subspace. Furthermore, we identified families of cycles as a fundamental structural feature enabling agreement and, leveraging this concept, characterized a class of graphs that support such protocols. Although our main conditions are structural and easy to check, they are only sufficient, and we infer that the communication graphs that are arbitrary k-agreement reachable is much larger in practice. This work opens several avenues for future research, including the use of nonlinear dynamics for achieving k-agreement, the consideration of packet drops and communication delays, the development of algorithms for distributed protocol synthesis, and the exploration of applications in distributed optimization. Moreover, closing the gap between the necessary and sufficient conditions remains an important direction for future investigation.

A Algebraic graph-theory

We summarize here some basic notions on algebraic graph theory from [8, 30] used throughout. A directed graph (or digraph) is $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. An edge $(i, j) \in \mathcal{E}$ means that node i can receive information [3] from j. Given $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, there is a one-to-one correspondence between A and a weighted digraph that has n nodes $\mathcal{V} = \{1, \ldots, n\}$ and a directed edge from j to i with edge weight a_{ij} when a_{ij} is nonzero and no edge otherwise. Conversely, given $\mathcal{G}$, we say that the matrix $A \in \mathbb{R}^{n\times n}$ is *consistent* with $\mathcal{G}$ if $(i, j) \notin \mathcal{E}$ implies $a_{ij} = 0$.

A *path* in $\mathcal{G}$ is a sequence of edges $(e_1, e_2, \ldots)$ such that the origin node of each edge is the destination node of the preceding edge. A graph is *strongly connected* if, for any $i, j \in \mathcal{V}$, there is a path from i to j. A graph is *complete* if there exists an edge connecting every pair of nodes, and it is *sparse* otherwise. A *closed path* is a path whose initial and final vertices coincide. A closed path is a *cycle* if, going along the path, one reaches no node, other than the initial-final node, more than once. A cycle of length equal to one is a

[3] Although the reverse convention (i.e., that $(i, j) \in \mathcal{E}$ means that node i can send information to j) could also be considered, this comes at the cost of replacing the adjacency matrix A by A^T. In order to avoid tedious transpose notation throughout, we have selected the stated convention.

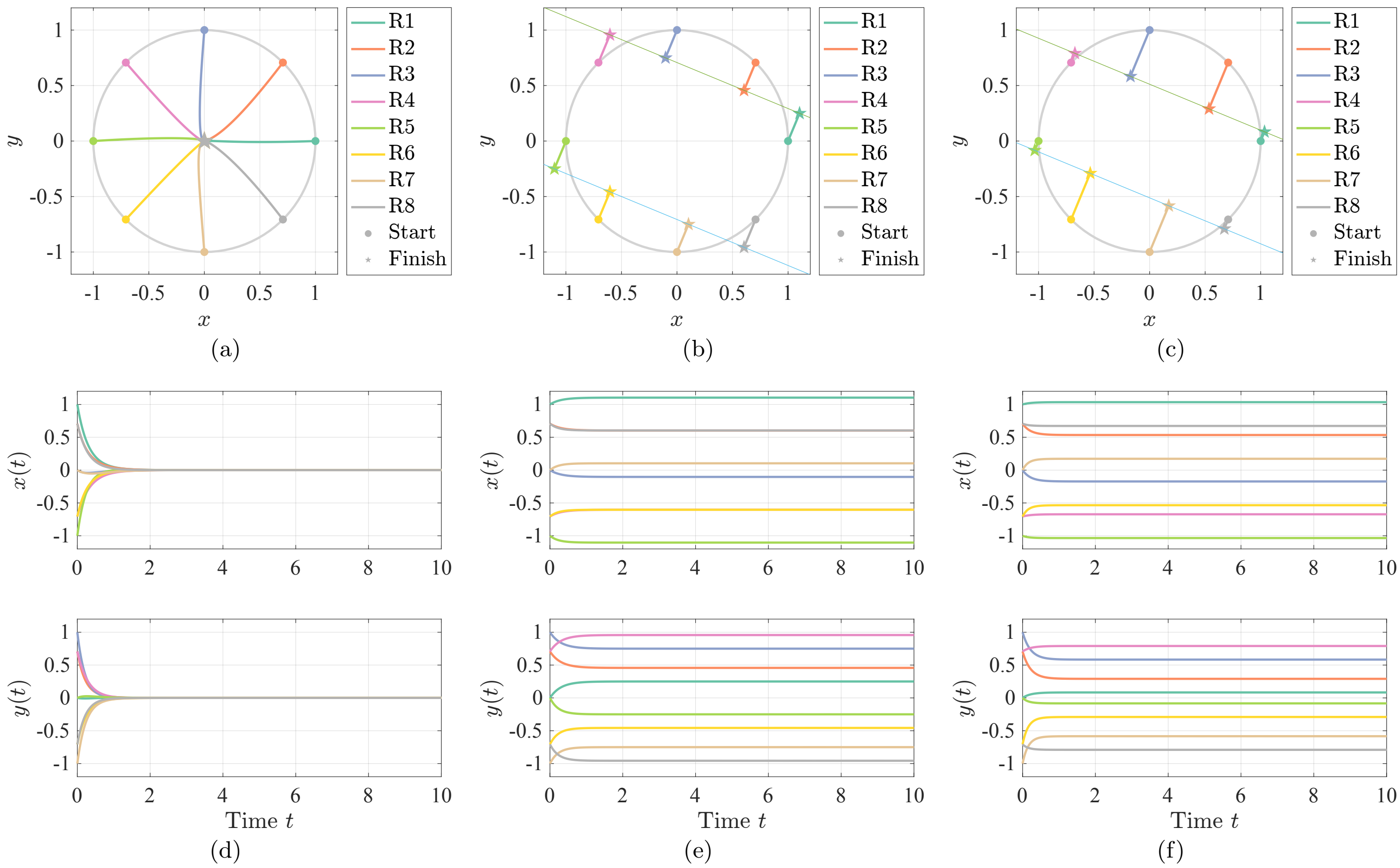

Fig. 4. (a)-(c) Time evolution of the positions of the 8 robots and (d)-(f) trajectories of the $x-$ and $y-$coordinates. (a) and (d) Consensus protocol, which allows the robots to achieve rendezvous. (b) and (e) k-agreement protocol on an orthogonal projection onto $\ker(M_1)$. (c) and (f) k-agreement on an oblique projection onto $\ker(M_1)$ along $\mathrm{Im}(N_1)$. See Section 5.

self cycle. A set of cycles that have no nodes in common is a *cycle family.* With a slight abuse of notation, we will denote a cycle family by $f = \{e_1, e_2, \dots\} \subseteq \mathcal{E}$, where $e_1, e_2, \dots \in \mathcal{E}$ are the edges involved in f. The *length of a cycle family* is the number of elements in the sequence $\{e_1, e_2, \dots\}$. We let $\mathcal{C}_\ell(\mathcal{G})$ denote the set of all cycle families of length ℓ in $\mathcal{G}$. See Fig. A.1 for an illustration. The *weight of a cycle family* is given by the product of the weights of all edges in the cycle family (namely, $\prod_{(i,j)\in f} a_{ij}$).

We will use a graph-theoretic interpretation of characteristic polynomials from [30].

Lemma A.1 ***([30, Thm. 1])*** *Let $\mathcal{G}$ be a digraph, A a matrix consistent with $\mathcal{G}$, and $\det(\lambda I - A) = \lambda^n + p_1\lambda^{n-1} + \dots + p_{n-1}\lambda + p_n$ its characteristic polynomial. Then, the coefficients $p_\ell, \ell \in \{1, \dots, n\}$, can be written as:*

$$p_\ell = \sum_{\xi\in\mathcal{C}_\ell(\mathcal{G})} (-1)^{n-d(\xi)} \prod_{(i,j)\in\xi} a_{ij},$$

where $d(\xi)$ is the number of cycles in the cycle family ξ. □

In other words, the ℓ-th coefficient of $\det(\lambda I - A)$ is given by a sum of terms, with each term given by the product of edge weights along a cycle family length ℓ.

Theorem A.2 ***([5, Thm. 5.3])*** *Let W be a matrix satisfying Assumption 1, let $t_1, \dots, t_n$ and $\tau_1, \dots, \tau_n$ be the vectors defined in (4). Additionally, let $\mathcal{G}$ be a graph satisfying Assumption 2, and $A \in \mathbb{R}^{n\times n}$ a matrix consistent with $\mathcal{G}$. The protocol $\dot{x} = Ax$ globally asymptotically reaches a k-agreement on W if and only if the following two conditions are satisfied:*

(i) $At_i = 0, \quad \tau_i^\mathsf{T} A = 0, \qquad \forall i \in \{1, \dots, k\}$.

(ii) The polynomial $\lambda^{n-k} + p_1\lambda^{n-k-1} + \dots + p_{n-k}$, with coefficients given by

$$p_\ell = \sum_{\xi\in\mathcal{C}_\ell(\mathcal{G})} (-1)^{d(\xi)} \prod_{(i,j)\in\xi} a_{ij}, \qquad \ell = 1, \dots, n-k,$$

is stable, where $d(\xi)$ denotes the number of cycles in the cycle family ξ. □

B Projections and linear subspaces

We summarize here some basic notions from linear algebra and geometry from [17] and [24, Sec.2]. Given a linear subspace $\mathcal{M} \subset \mathbb{R}^n$, its *orthogonal complement* is $\mathcal{M}^\perp := \{x \in \mathbb{R}^n \;:\; x^\mathsf{T} y = 0, \forall\, y \in \mathcal{M}\}$. Given two subspaces $\mathcal{M}, \mathcal{N} \subseteq \mathbb{R}^n$, $\mathcal{M}\cap\mathcal{N} = \{0\}$, their *direct sum* is $\mathcal{W} := \{u+v \;:\; u \in \mathcal{M}, v \in \mathcal{N}\}$ and denoted by $\mathcal{W} = \mathcal{M} \oplus \mathcal{N}$; $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$ are *complementary* if $\mathcal{M} \oplus \mathcal{N} = \mathbb{R}^n$. Given complementary subspaces $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$, for any $z \in \mathbb{R}^n$, there exists a unique decomposition $z = x + y$, where $x \in \mathcal{M}$ and $y \in \mathcal{N}$. The transformation $\Pi_{\mathcal{M},\mathcal{N}}$, defined by $\Pi_{\mathcal{M},\mathcal{N}} z := x$, is called *projection onto $\mathcal{M}$ along $\mathcal{N}$*, and the transformation $\Pi_{\mathcal{N},\mathcal{M}}$ defined by $\Pi_{\mathcal{N},\mathcal{M}} z := y$ is called *projection onto $\mathcal{N}$ along $\mathcal{M}$*. Vector x is the projection of z onto $\mathcal{M}$ along $\mathcal{N}$, and y

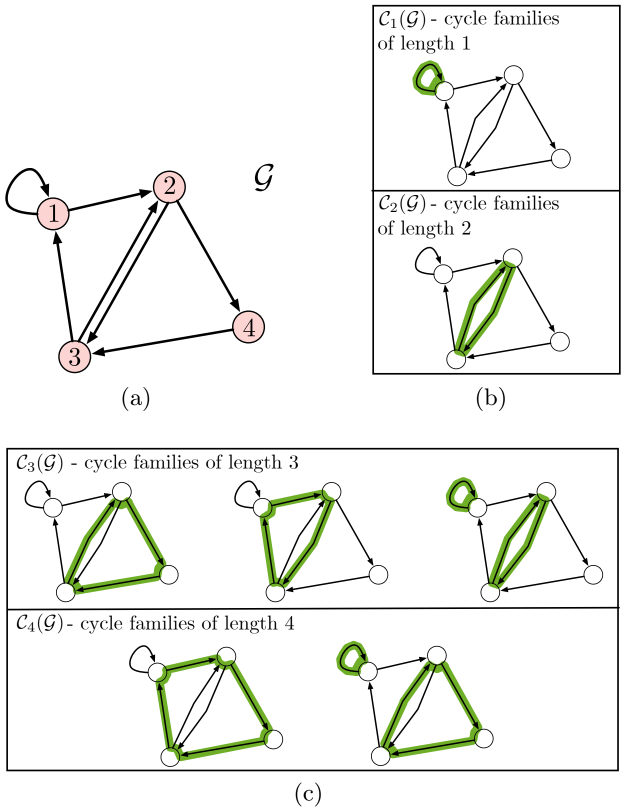

Fig. A.1. (a) Illustration of a directed graph, and (b)–(c) corresponding cycle families, grouped by cycle length. Note that this graph contains one cycle family of length $\ell = 1$, one of length $\ell = 2$, three of length $\ell = 3$, and two of length $\ell = 4$.

is the projection of z onto $\mathcal{N}$ along $\mathcal{M}$. A matrix $\Pi \in \mathbb{R}^{n\times n}$ is a projection onto some subspace if and only if $\Pi^2 = \Pi$. The projection $\Pi_{\mathcal{M},\mathcal{M}^\perp}$ onto $\mathcal{M}$ along $\mathcal{M}^\perp$ is called *orthogonal projection onto* $\mathcal{M}$. Because the subspace $\mathcal{M}$ uniquely determines $\mathcal{M}^\perp$, we will denote $\Pi_{\mathcal{M},\mathcal{M}^\perp}$ compactly as $\Pi_{\mathcal{M}}$. Projections that are not orthogonal are called *oblique*.

Lemma B.1 ***[17, Thm. 2.11 and Thm. 2.31]*** *Let $\Pi \in \mathbb{R}^{n\times n}$ be a projection with* $\operatorname{rank}(\Pi) = k$. *There exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that*

$$\Pi = T \begin{bmatrix} I_k & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} T^{-1}.$$

Moreover, if Π is an orthogonal projection, then T can be chosen to be an orthogonal matrix, i.e., $TT^\mathsf{T} = I$. □

Lemma B.2 ***[17, Thm. 2.26]*** *Let $\mathcal{M}, \mathcal{N}$ be complementary subspaces and the columns of $M \in \mathbb{R}^{n\times k}$ and $N \in \mathbb{R}^{n\times k}$ form a basis for $\mathcal{M}$ and $\mathcal{N}^\perp$, respectively. Then,*

$$\Pi_{\mathcal{M},\mathcal{N}} = M(N^\mathsf{T} M)^{-1} N^\mathsf{T}.$$ □

We recall the following known properties [17, Thm. 1.60]:

$$\operatorname{Im}(M^\mathsf{T}) = \operatorname{Im}(M^\dagger) = \operatorname{Im}(M^\dagger M) = \operatorname{Im}(M^\mathsf{T} M),$$
$$\ker(M) = \operatorname{Im}(M^\mathsf{T})^\perp = \ker(M^\dagger M) = \operatorname{Im}(I - M^\dagger M).$$

From these properties and Lemma B.2, given $M \in \mathbb{R}^{m\times n}$:

$$\Pi_{\operatorname{Im}(M)} = MM^\dagger, \qquad \Pi_{\ker(M)} = I - M^\dagger M,$$

where $M^\dagger \in \mathbb{R}^{n\times m}$ is the Moore-Penrose inverse of M.

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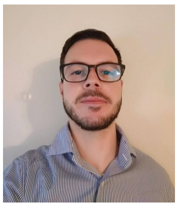

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