Cycle families and Resilience of Dynamical Networks

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***Abstract*— Dynamical network models are a flexible framework to describe groups of dynamical systems interacting through a network and have been widely used in several applications to model real-world systems, including transportation, communication, and biology. In this paper, we investigate the resilience of dynamical network models under structured perturbations of their edges. Given a linear dynamical network with the property that poles are confined to a prescribed region, we ask whether it is possible to compromise this property by perturbing a single communication edge. We prove that only a subset of the edges, if perturbed, could compromise stability and we provide a graph-theoretic characterization to determine these edges. Interestingly, our results show that only edges that belong to some cycles of the communication graph play a role in the considered measure of resilience, thus identifying cycles as the basic element that determines resilience in dynamical networks. The theoretical guarantees are illustrated through simulations applied to a nonlinear epidemic model.**

I. Introduction

Dynamical network models are widely used across several scientific domains to describe real-world systems organized as groups of disjoint components operating in synchrony through a network interconnection. In many cases, systems are large-scale and entail massive numbers of components whose behavior may vary as a result of failures or external attack actions. Accordingly, a critical property of dynamical networks is their resilience, which is a property that guarantees that systems can continue to operate effectively in the face of perturbations. Despite the importance of ensuring resilience in the applications, many established control-theoretic properties of dynamical network models – such as system stability, controllabity, etc. – are defined as binary notions [1]; namely, a network is either stable or unstable, controllable or uncontrollable, and so on. Unfortunately, a binary notion fails to describe how robust a certain property is. For example, a stable system may or may not be "close" to instability [2], and thus it is of central importance to study measures that can quantify the robustness of these properties.

In this paper, we focus on dynamical network models with linear dynamics, and we investigate their resilience measured as the smallest number of edges that need to be perturbed in order to shift the poles of the associated dynamical system outside of a nominal region. Obtaining such characterizations is difficult for two reasons: pole locations are nonlinear functions of the edge parameters (thus making it challenging to relate how perturbations of the edges will affect the poles of the entire network) and, second, because our metric of interest is the L_0-(pseudo)norm (which is known to be non-convex and non-differentiable), the associated decision problem is challenging to solve numerically. Here, we overcome these challenges by taking a graph-theoretic approach. Interestingly, the graph-theoretic approach allows us to provide some insights into the solution of a problem that has been shown to be NP-hard [3] in the general case.

Related work. We model networks using a structured linear time-invariant model and analyze its pole locations in the face of perturbations to the entries of the adjacency matrix; this connects our work to the stability radius of linear systems [4], and the closely-related controllability radius [5]. When the perturbation is allowed to be complex and there are no structural constraints, [5] showed that the controllability radius can be characterized using singular value decompositions and a polynomial-time algorithm. When the perturbation is restricted to be real, an algorithm is proposed in [2]. For the stability radius, early studies also allowed complex and unstructured perturbations [4], [6]; real but unstructured perturbations were considered in [7]–[9].

The case of real and structured perturbations has been tackled only more recently. Structured perturbations of the form $\Delta = BDC$ are considered in [10] and a tractable solution has been provided in [11]. According to [10], assuming $\Delta = BDC$ is rather simplistic, as it does not allow to account for cases where a group of entries of Δ must be identically zero or certain entries are coupled with each other. Motivated by this, [12], [13] present first-order optimality conditions and iterative algorithms to determine observability and stability radii, respectively. General affine perturbations of the form $\Delta = \sum_i \theta_i A_i$ have been studied recently in [14], which provides necessary conditions for local optimality and an algorithm to determine locally optimal points. The authors of [15], [16] further refined these techniques using relaxations from convex optimization, and tractable reformulations are given in [17]. We remark that the L_0-norm stability radius was considered in [16], which proved the hardness of the problem and derived heuristic algorithms and bounds. Network resilience against edge modifications has recently been considered in [18] by taking a frequency-domain approach. Finally, the importance of cycles in the stability of networks has been observed also in [19], [20].

Contributions. The contribution of this paper is threefold. First, we formulate a resilience question for dynamical network models, which asks to determine which edges, if perturbed, could lead to the deterioration of a nominal system property (pole locations). We provide a graph-theoretic characterization of all admissible perturbations that could deteriorate such a property, and we show that only edges that belong to some cycles of the graph topology could compro-

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mise the property of interest. Second, we provide a closed-form expression for the perturbation of interest. Third, we investigate the resilience properties of some graph topologies commonly adopted in practice. Finally, we demonstrate the applicability of the framework through a set of simulations[1].

II. Preliminaries

We introduce here some instrumental notions.

A *weighted digraph* $\mathcal{G}_A := (\mathcal{V}, \mathcal{E}, A)$ consists of a set of nodes $\mathcal{V}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ (an element $(i,j) \in \mathcal{E}$ denotes a directed edge from node j to i), and an *adjacency matrix* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, where a_{ij} describes the *edge weight* of (i,j). A *path* in $\mathcal{G}_A$ is a sequence of edges $\{e_1, e_2, \dots\}$ such that the initial node of each edge coincides with the final node of the preceding one. The *length* of a path is the number of edges in the sequence $\{e_1, e_2, \dots\}$. A *closed path* is a path whose initial and final nodes coincide; a closed path is a *cycle* if, going along the path, one reaches no node other than the first and last more than once; a cycle of length equal to one is a *self-cycle*. We call a set of node-disjoint cycles a *cycle family*; with a slight abuse of notation, it will be convenient to denote a cycle family by $f = \{e_1, e_2, \dots\}$, where $e_1, e_2, \dots,$ are the edges involved in all the cycles that form f. The length of a cycle family is the number of edges in the sequence $\{e_1, e_2, \dots\}$. The *weight* of a cycle family f, denoted by w_f, is given by the product of the weights of all edges involved; namely, $w_f := \prod_{(i,j)\in f} a_{ij}$. See Fig. 1 for an illustration.

Let $\mathcal{F}_\ell, \ell \in \{1, \dots, n\}$, be the set of all cycle families of length ℓ. The following result is instrumental.

Lemma 2.1: ***([21, Thm. A2.5])*** Consider the digraph $\mathcal{G}_A$, and let the characteristic polynomial of A be:

$$\det(\lambda I - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n.$$

Each coefficient $p_\ell, \ell \in \{1, \dots, n\}$, can be expressed as:

$$p_\ell = \sum_{f\in\mathcal{F}_\ell} (\text{-}1)^{n-d_f} w_f, \tag{1}$$

where d_f denotes the number of disjoint cycles in f. □

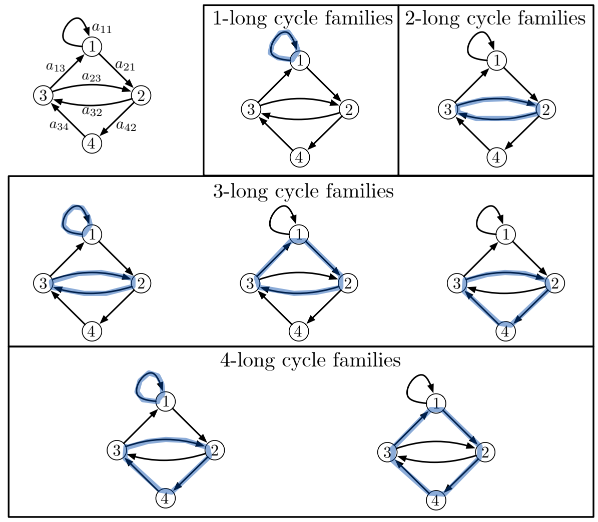
The lemma provides a graph-theoretic expression for the characteristic polynomial of A: it shows that the ℓ-th coefficient can be computed as the sum of the weights of all cycle families of length ℓ in $\mathcal{G}_A$, with adjusted sign.

III. Problem formulation

We consider dynamical networks whose communication topology can be described by a weighted digraph $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}, A)$ (see Section II). In what follows, we let $\mathcal{V} = \{1, \dots, n\}$. Each element $i \in \mathcal{V}$ models an agent whose state is a scalar variable $x_i \in \mathbb{R}$, and the joint network state $x = (x_1, \dots, x_n)$ updates according to:

$$\dot{x} = Ax. \tag{2}$$

[1]*Notation.* $\|x\|_p$ denotes the L_p-norm (or p-norm for short) of $x \in \mathbb{R}^n$. For $\lambda \in \mathbb{C}$, $\text{Re}(\lambda)$ and $\text{Im}(\lambda)$ denote its real and imaginary parts, respectively. For $A \in \mathbb{R}^{n\times n}$, $\sigma(A)$ denotes its spectrum and $\det(A)$ its determinant. $|\mathcal{F}|$ denotes the cardinality of set $\mathcal{F}$.



(a)

Fig. 1: Illustration of the adopted notation and graph-theoretic notions. (top-left) Example of a weighted digraph with 4 nodes and 7 edges; (all other figures) associated cycle families.

To be consistent with the network structure imposed by $\mathcal{G}_A$, $(i,j) \notin \mathcal{E}$ implies $a_{ij} = 0$, describing that some states do not have a direct action on others. We utilize (2) to describe the nominal (unperturbed) network and we assume that such a system is well-behaved in the following sense[2].

Assumption 1: ***(Nominal pole locations)*** Let $\mathcal{D} \subset \mathbb{C}$ be an open set such that $\partial\mathcal{D} \cap \mathbb{R} \neq \emptyset$. All eigenvalues of A belong to $\mathcal{D}$, namely, $\text{rank}(A - \lambda I) = n, \forall \lambda \in \mathcal{D}^c := \mathbb{C} \setminus \mathcal{D}$.

Assumption 1 guarantees that all the poles of (2) belong to a prescribed region of the complex plane. The technical condition $\partial\mathcal{D} \cap \mathbb{R} \neq \emptyset$ ensures that the boundary of $\mathcal{D}$ (or, equivalently, of $\mathcal{D}^c$ since $\partial\mathcal{D} = \partial\mathcal{D}^c$) contains some element of the real axis. This assumption holds for most sets $\mathcal{D}$ of practical interest, as discussed in the following example.

Example 1: When

$$\mathcal{D} = \{\lambda \in \mathbb{C} \,:\, \text{Re}(\lambda) < 0\}, \tag{3}$$

Assumption 1 asserts that the nominal system belongs to the class of asymptotically stable linear systems. Alternatively, for fixed $\zeta > 0$, systems that satisfy Assumption 1 with

$$\mathcal{D} = \{\lambda \in \mathbb{C} : \text{Re}(\lambda) < -\omega\zeta,$$
$$|\text{Im}(\lambda)| < \omega\sqrt{1-\zeta^2} \text{ for some } \omega \in \mathbb{R}\},$$

are systems whose damping ratio is upper bounded by ζ. For given $\lambda_c \in \mathbb{C}$ and $r \in \mathbb{R}$, systems with

$$\mathcal{D} = \{\lambda \in \mathbb{C} \,:\, |\lambda - \lambda_c| < r\},$$

have all poles inside a disc centered at λ_c with radius r. □

In this work, we consider perturbations of the edge weights of $\mathcal{G}_A$ that seek to shift the poles of (2) outside of the nominal region $\mathcal{D}$. We focus on structured perturbations, in the sense that only certain edge weights are allowed to be perturbed. To account for structured perturbations, let $\mathcal{E}_\Delta \subseteq \mathcal{E}$ denote

[2]For $\mathcal{D} \subseteq \mathbb{C}$, we denote by $\partial\mathcal{D}$ its boundary.

the set of edges that are allowed to be perturbed, and let the set of matrices compatible with $\mathcal{E}_\Delta$ be:

$$\mathcal{S} = \{S = [s_{ij}] \in \mathbb{R}^{n\times n} : s_{ij} = 0 \text{ if } (i,j) \notin \mathcal{E}_\Delta\}.$$

For example, the condition $\mathcal{E}_\Delta \subseteq \mathcal{E}$ ensures that only existing edges can be perturbed and no edge can be added. We thus consider perturbation matrices $\Delta = [d_{ij}]$ such that $\Delta \in \mathcal{S}$. Subject to perturbations, the dynamics (2) are modified to:

$$\dot{x} = (A + \Delta)x. \tag{4}$$

The problem focus of this work is formalized next.

Problem 1: ***(Problem of interest)*** Find, when possible, a structured perturbation $\Delta \in \mathcal{S}$ of a single edge (i.e., $\|\Delta\|_0 = 1$) such that the poles of the perturbed system (4) are no longer confined within the nominal region $\mathcal{D}$. □

In the remainder, we will use the following notions.

Definition 1: ***(Admissible perturbations)*** The matrix $\Delta \in \mathcal{S}$ is said to be an *admissible perturbation* if

$$\sigma(A + \Delta) \cap \mathcal{D}^c \neq \emptyset. \tag{5}$$

Further, $\Delta \in \mathcal{S}$ that satisfies (5) and such that $\|\Delta\|_0 = 1$ will be called a *single-edge admissible perturbation.* □

Before proceeding, we discuss an important relationship between Problem 1 and the 0-norm stability radius [16]. The 0-norm stability radius amounts to determining solutions to the following optimization problem:

$$\begin{aligned} \min_{\Delta\in\mathcal{S}} \quad & \|\Delta\|_0 \\ \text{subject to:} \quad & \sigma(A + \Delta) \cap \mathcal{D}^c \neq \emptyset. \end{aligned} \tag{6}$$

It is immediate to see that single-edge admissible perturbations considered here are solutions of the optimization problem (6). On the other hand, a solution of (6) is a single-edge admissible perturbations only if $\|\Delta\|_0 = 1$.

The formulation (6) highlights some of the challenges related to solving Problem 1. First, since the cost function in (6) involves the L_0-pseudonorm – which is a non-convex and non-differentiable function [22] – classical optimization techniques cannot be utilized to determine globally optimal solutions. Second, since the constraint in (6) is nonlinear and closed-form expressions are challenging to obtain, seeking numerical solutions to (6) is out of reach in general. Finally, we remark that [16] showed that (a slightly more general version of) (6) is equivalent to an NP-hard decision problem. A similar observation was also made in [23]. These findings illustrate the challenges in tackling Problem 1.

IV. Graph-theoretic characterization of single-edge admissible perturbations

In this section, we tackle Problem 1 by taking a graph-theoretic approach. The following is our first main result.

Theorem 4.1: ***(Existence of admissible perturbations)*** Let Assumption 1 hold and let $\mathcal{F}_\ell$, denote the set of ℓ-long cycle families in $\mathcal{G}_A$. There exists an admissible perturbation if and only if

$$\mathcal{F}_\ell \cap \mathcal{E}_\Delta \neq \emptyset, \tag{7}$$

for some $\ell \in \{1, \dots, n\}$. □

Proof: We begin by noting that, since $\|\Delta\|_0$ is lower semicontinuous [22], if Δ is a single-edge admissible perturbation, then, there exists $\tilde{\Delta} \in \mathcal{S}$ such that $\|\tilde{\Delta}\|_0 = 1$ and $\sigma(A + \tilde{\Delta}) \cap \partial\mathcal{D} \neq \emptyset$. In other words, the search for single-edge admissible perturbations can be restricted by seeking intersections with the boundary $\partial\mathcal{D}$. Accordingly, in what follows we restrict our focus to seeking perturbations such that $\det(\lambda I - A - \Delta) = 0$ with $\lambda \in \partial\mathcal{D}$. Hence, by denoting by $\{p_1, \dots, p_n\}$ the coefficients of the characteristic polynomial of $A + \Delta$, the statement of the theorem can be restated as follows: there exists $\lambda \in \partial\mathcal{D}$ such that

$$\begin{aligned} 0 &= \det(\lambda I - A - \Delta) \\ &= \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n, \end{aligned} \tag{8}$$

if and only if (7) holds. We now proceed to prove this claim.

(If) By interpreting the coefficients $\{p_1, \dots, p_n\}$ graph-theoretically as in (1), when (7) holds we conclude that at least one of the coefficients $\{p_1, \dots, p_n\}$ can be modified arbitrarily by perturbing a single edge in $\mathcal{E}_\Delta$. Hence, by letting $e = (k, p) \in \mathcal{E}_\Delta$ be an arbitrary edge that satisfies (7) (if multiple such edges exists, e can be chosen arbitrarily), by setting $d_{ij} = 0$ for all $(i, j) \in \mathcal{E}_\Delta \setminus \{e\}$, and by letting $\lambda \in \mathbb{R} \cap \partial\mathcal{D}$ (cf. Assumption 1), we obtain that (8) is a linear equation in the unknown d_{kp}. Notice that the coefficient multiplying d_{kp} is guaranteed to be nonzero: by contradiction, if such coefficient were equal to zero, then (k, p) would not belong to a cycle family. To conclude, by the Rouché–Capelli theorem, there exists d_{kp} such that (8) holds.

(Only if) Assume, by contradiction, that $\mathcal{F}_\ell \cap \mathcal{E}_\Delta = \emptyset$ and that (8) holds. By application of (1), all coefficients $\{p_1, \dots, p_n\}$ are independent of Δ, and thus $\det(\lambda I - A - \Delta) = \det(\lambda I - A)$. But this is a contradiction, since by Assumption 1 the roots of the characteristic polynomial of A are confined in $\mathcal{D}$. ■

Theorem 4.1 provides a graph-theoretic condition to check existence of admissible perturbations: a single-edge admissible perturbation exists if and only if the perturbation set $\mathcal{E}_\Delta$ contains at least one edge that appears in some cycle family of $\mathcal{G}_A$. To the best of our knowledge, this is the first graph-theoretic characterization of the feasibility of Problem 1.

Theorem 4.1 allows us to identify which edges of the graph, if perturbed, could compromise the nominal pole locations formalized in Assumption 1. We illustrate the applicability of this finding in the following example.

Example 2: Consider the graph $\mathcal{G}_A$ illustrated in Fig. 2(a). Fig. 2(b) illustrates all cycle families of this graph. By inspection, it is immediate to see that the edges $\{a_{13}, a_{12}, a_{24}\}$ do not appear in any cycle family of this graph. By application of Theorem 4.1, we conclude that any perturbation of these edges will not shift the poles outside of a nominal region $\mathcal{D}$, no matter how $\mathcal{D}$ is defined. Conversely, the theorem guarantees existence of a perturbation of each of the remaining edges $\{a_{11}, a_{22}, a_{33}, a_{43}, a_{34}\}$ that shifts the eigenvalues outside of $\mathcal{D}$. □

Remark 1: ***(Complexity of checking*** (7)***)*** By observing that every cycle of $\mathcal{G}_A$ appears in some cycle family of the

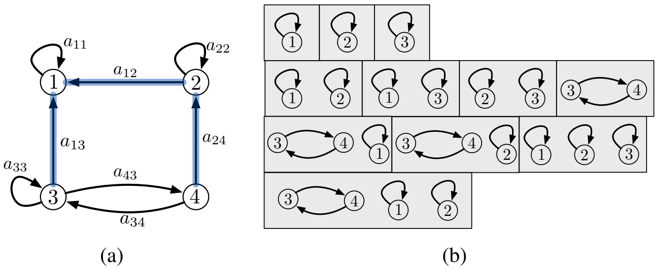


Fig. 2: (a) Communication topology discussed in Example 2. (b) Associated cycle families. By application of Theorem 4.1, any perturbation of the edges a_{13}, a_{12}, a_{24} will not shift the poles outside of $\mathcal{D}$. Conversely, there exists a perturbation of each of the remaining edges $\{a_{11}, a_{22}, a_{33}, a_{43}, a_{34}\}$, that will shift the eigenvalues outside of $\mathcal{D}$. See Example 2.

graph (indeed, a cycle of length ℓ is itself an ℓ-long cycle family), it follows that checking (7) is equivalent to checking whether some element of $\mathcal{E}_\Delta$ appears in some cycle of $\mathcal{G}_A$. Further, by noting that determining all elementary cycles in a graph can be done in polynomial time (of $|\mathcal{V}|$ and $|\mathcal{E}|$) using the algorithm of Johnson [24], it follows that evaluating (7) can also be done in polynomial time. □

V. Computing single-edge admissible perturbations

While Theorem 4.1 provides a way to check existence of admissible perturbations, it remains to address the question of how to determine such perturbations, when they exist. This is the goal of this section. In what follows, we will denote by $e_i \in \mathbb{R}^n$ the i-th canonical vector. For given $(i,j) \in \mathcal{V}$, we will consider single-edge perturbations of the form:

$$\Delta := d e_i e_j^{\mathsf{T}}, \tag{9}$$

which correspond to a perturbation of magnitude d of the (i,j)-th entry of A. The following is our second main result.

Theorem 5.1: ***(Characterization of single-edge admissible perturbations)*** Let Assumption 1 hold and $(i,j) \in \mathcal{E}$. There exists $d \in \mathbb{C}$ such that $\Delta = d e_i e_j^{\mathsf{T}}$ satisfies

$$\sigma(A+\Delta) \cap \partial\mathcal{D} \neq \emptyset, \tag{10}$$

if and only if for some $\lambda \in \partial\mathcal{D}$

$$e_j^{\mathsf{T}}(A-\lambda I)^{-1} e_i \neq 0. \tag{11}$$

Moreover, when (11) holds,

$$d = -\frac{1}{e_j^{\mathsf{T}}(A-\lambda I)^{-1} e_i}, \tag{12}$$

is the unique choice of $d \in \mathbb{C}$ such that (9) satisfies $\lambda \in \sigma(A+\Delta)$.

Proof: By application of the Woodbury matrix identity[3]:

$$\begin{aligned}(A-\lambda I + e_i d e_j^{\mathsf{T}})^{-1} &= (A-\lambda I)^{-1} \\ &- (A-\lambda I)^{-1} e_i (d^{-1} + e_j^{\mathsf{T}}(A-\lambda I)^{-1} e_i)^{-1} e_j^{\mathsf{T}} (A-\lambda I)^{-1}.\end{aligned}$$

[3]The Woodbury matrix identity states that $(F+UCV)^{-1} = F^{-1} - F^{-1}U(C^{-1}+VF^{-1}U)VF^{-1}$ for matrices F, U, C, V of suitable sizes.

Since $(A-\lambda I)$ is invertible by Assumption 1, and by noting that $(d^{-1} + e_j^{\mathsf{T}}(A-\lambda I)^{-1}e_i)^{-1}$ is a scalar quantity, we conclude that $\det(A+\Delta-\lambda I) = 0$ holds if and only if

$$d^{-1} + e_j^{\mathsf{T}}(A-\lambda I)^{-1} e_i = 0. \tag{13}$$

To prove the first part of the claim, notice that $e_j^{\mathsf{T}}(A-\lambda I)^{-1}e_i = 0$ in equation (13) yields $d^{-1} = 0$. The second claim follows by noting that (13) is equivalent to (12). ■

The theorem provides in (12) an explicit formula to compute perturbations that shift the eigenvalues to the boundary of the stability region. In words, the unique perturbation of the (i,j)-th entry of A that makes $\lambda \in \sigma(A + d e_i e_j^{\mathsf{T}})$ is given by the negative inverse of the (j,i)-th entry of $(A-\lambda I)^{-1}$. This result is useful because the application of (12) with some choice of $\lambda \in \mathcal{D}^c$ guarantees that $\sigma(A + d e_i e_j^{\mathsf{T}}) \cap \mathcal{D}^c \neq \emptyset$, thus providing a closed-form expression for perturbations that shift the eigenvalues outside of $\mathcal{D}$. We remark that, although Δ computed using (12) may be a complex matrix in general (when $\lambda \in \mathbb{C}$), by Assumption 1, there exists $\lambda \in \mathbb{R}$ on the boundary $\partial\mathcal{D}$, and thus this choice of λ in (12) yields a real perturbation. We also remark that when one seeks single-edge admissible perturbations that, further, are of minimal 2-norm, these are obtained by choosing $\lambda \in \partial\mathcal{D}$ in (12). This follows since the 2-norm is a monotone function of the absolute value of its entries.

Altogether, Theorems 4.1 and 5.1 provide a complete, tractable, answer to Problem 1. We conclude by noting that, our result recovers the expression for the stability radius given in [11, eq. (2)] as a special case.

VI. Application to study the robustness of equilibrium points in epidemic models

In this section, we illustrate the applicability of the framework to mitigate the spread of a disease modeled through an SIS (Susceptible-Infected-Susceptible) model of epidemic spread. Consider a population affected by the spread of a certain disease, and assume that this population can be organized into n sub-groups $\mathcal{V} = \{1, \dots, n\}$, each describing an epidemiologically homogeneous population. To every group $g \in \mathcal{V}$, we associate two time-dependent state variables to describe the epidemic state: $i_g : \mathbb{R}_{\geq 0} \to [0,1]$ (fraction of infected population) and $s_g : \mathbb{R}_{\geq 0} \to [0,1]$ (fraction of susceptible population). We adopt an SIS model with constant population [26] to describe the epidemic spread in each region, described by:

$$\begin{aligned}\frac{d}{dt} s_g &= -\beta_g s_g \sum_{\ell \in \mathcal{V}} b_{g\ell} i_\ell + \gamma_g i_g, \\ \frac{d}{dt} i_g &= \beta_g s_g \sum_{\ell \in \mathcal{V}} b_{g\ell} i_\ell - \gamma_r i_g, \qquad \forall\, g \in \mathcal{V},\end{aligned} \tag{14}$$

where, for each $g \in \mathcal{V}$, $\beta_g > 0$ is the transmission rate, $\gamma_g > 0$ is the recovery rate, and $b_{g\ell} > 0$ models the intensity of infection due to interactions between susceptibles from group g and infected individuals from group ℓ. Since the population size is constant, $s_g = 1 - i_g\ \forall\, g \in \mathcal{V}$ at all times, and thus the model is fully characterized by the infected state.

TABLE I: Parameter values associated with COVID-19 data on 06/16/21 in the state of Colorado, USA, taken from [25]. The recovery rate is $\gamma = 1/9$ (days^{-1}). Transmission rates β (days^{-1}) are derived by multiplying the effective reproduction number in [25] by the recovery rate.

$$B = \begin{bmatrix} 0.0738 & 0 & 0 & 0.0022 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0024 & 0.0688 & 0 & 0.0130 & 0.0016 & 0.0021 & 0 & 0 & 0 & 0 & 0 \\ 0.0025 & 0 & 0.0639 & 0.0082 & 0.0019 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0635 & 0.0017 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0048 & 0.0722 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0017 & 0 & 0.0035 & 0.0012 & 0.1132 & 0 & 0 & 0 & 0 & 0.0016 \\ 0.0027 & 0 & 0 & 0.0044 & 0.0012 & 0 & 0.1094 & 0.0020 & 0 & 0 & 0 \\ 0.0036 & 0 & 0 & 0.0021 & 0 & 0 & 0 & 0.0921 & 0.0011 & 0 & 0 \\ 0.0014 & 0 & 0 & 0.0018 & 0.0012 & 0 & 0 & 0.0032 & 0.1022 & 0 & 0 \\ 0 & 0 & 0 & 0.0018 & 0 & 0 & 0 & 0 & 0 & 0.0729 & 0 \\ 0.0015 & 0 & 0 & 0.0047 & 0 & 0.0041 & 0 & 0 & 0 & 0.0010 & 0.1081 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0.0738 \\ 0.0688 \\ 0.0639 \\ 0.0635 \\ 0.0722 \\ 0.1132 \\ 0.1094 \\ 0.0921 \\ 0.1022 \\ 0.0729 \\ 0.1081 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0.0076 \\ 0.0010 \\ 0.0109 \\ 0.0060 \\ 0.0062 \\ 0.0182 \\ 0.0103 \\ 0.0104 \\ 0.0137 \\ 0.0056 \\ 0.0083 \end{bmatrix}.$$

Accordingly, in what follows we will drop the first equation in (14). In vector form, the infected states of (14) read as:

$$\dot{x} = (I - \operatorname{diag}(x)) \operatorname{diag}(\beta) B x - \operatorname{diag}(\gamma) x, \qquad (15)$$

where $x = (i_1, \ldots, i_n)$ is the vector of infected states, $B = [b_{r\ell}] \in \mathbb{R}^{n \times n}$ is the matrix of interactions, $\beta = (\beta_1, \ldots, \beta_n)$ is the vector of transmission rates, $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the vector of recovery rates, and, for a vector $v \in \mathbb{R}^n$, $\operatorname{diag}(v) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with entries given by v.

Before we can formalize the problem of interest, it is necessary to recall some well-known properties of SIS models. First, it is well-known [26] that $[0,1]^n$ is positively invariant for (15). For $M \in \mathbb{R}^{n \times n}$, let $\rho(M) := \max\{|\lambda| : \lambda \in \sigma(M)\}$ denote its spectral radius. The *basic reproduction number* of (15) is:

$$\mathcal{R}_0(B, \beta, \gamma) := \rho(\operatorname{diag}(\gamma)^{-1} \operatorname{diag}(\beta) B),$$

and $\operatorname{diag}(\gamma)^{-1} \operatorname{diag}(\beta) B$ is the *next generation matrix*. Because all entries of γ, β, A are non-negative, $\mathcal{R}_0(B, \beta, \gamma) \geq 0$. For $x \in \mathbb{R}^n$, we will write $x > 0$ to denote that all its entries are positive; we will say that $M \in \mathbb{R}^{n \times n}$ is nonnegative if all its entries are nonnegative; a nonnegative matrix M is irreducible if, for any nontrivial partition J, K of the index set $\{1, \ldots, n\}$, there exists $j \in J$ and $k \in K$ such that $a_{jk} \neq 0$.

Proposition 6.1: (see [26]) Assume $\operatorname{diag}(\gamma)^{-1} \operatorname{diag}(\beta) A$ is non-negative and irreducible. Then, (15) admits a unique equilibrium point. Moreover, if $\mathcal{R}_0(B, \beta, \gamma) \leq 1$, the unique equilibrium is $x = 0$ and, if $\mathcal{R}_0(B, \beta, \gamma) > 1$, the unique equilibrium satisfies $x > 0$. □

Intuitively, irreducibility of $\operatorname{diag}(\gamma)^{-1} \operatorname{diag}(\beta) A$ means that no population group is isolated from the rest. As commonly done in the literature, we will call the equilibrium $x = 0$ *disease-free,* and an equilibrium $x > 0$ *endemic.*

Proposition 6.2: [26, Thm.s 2.3 and 2.4] Assume that $\operatorname{diag}(\gamma)^{-1} \operatorname{diag}(\beta) A$ is non-negative and irreducible. Then, the following claims hold for (15).

(a) When the disease-free equilibrium exists, it is globally asymptotically stable.

(b) When the endemic equilibrium exists, it is locally asymptotically stable everywhere in $[0,1]^n \setminus \{0\}$. □

It follows from Propositions 6.1-6.2 that the endemic properties of the disease are fully characterized by $\mathcal{R}_0(B, \beta, \gamma)$. Precisely, if $\mathcal{R}_0(B, \beta, \gamma) \leq 1$, from any initial condition in $[0,1]^n$, the dynamics will converge to the disease-free equilibrium; on the other hand, if $\mathcal{R}_0(B, \beta, \gamma) > 1$, from any initial point in $[0,1]^n \setminus \{0\}$, the dynamics will converge to an endemic equilibrium. With this background, in what follows we will take the perspective of a policymaker and tackle the problem of finding minimal interventions the be applied to (15) that will lead to an eradication of the disease. We formalize this objective next.

Problem 2: ***(Disease eradication)*** Given a disease as in (15), which satisfies $\mathcal{R}_0(B, \beta, \gamma) > 1$, determine a minimal perturbation ΔB of the existing inter-group infection intensities B that yields $\mathcal{R}_0(B + \Delta B, \beta, \gamma) \leq 1$. □

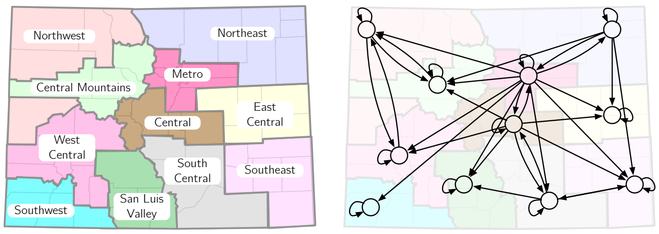
We remark that although $\mathcal{R}_0$ can be adapted by modifying any of the parameters β, γ, B, in what follows we focus on modifications of B only, which correspond to interventions that modify the interaction intensities between the groups. Problem 2 is an instance of Problem 1 with $\mathcal{E}_\Delta = \mathcal{E}$ and

$$\mathcal{D} = \{\lambda \in \mathbb{C} \,:\, \rho(\lambda) > 1\}, \quad A \;= \operatorname{diag}(\gamma)^{-1} \operatorname{diag}(\beta) B.$$

Moreover, one can recover solutions to Problem 2 from those of Problem 1 using:

$$\Delta B = \operatorname{diag}(\beta)^{-1} \operatorname{diag}(\gamma) \Delta, \qquad (16)$$

which is always well-defined since $\beta > 0$. For our simulations, we use model parameters describing the COVID-19 disease on 06/16/21 in the state of Colorado, USA, publicly available at [25]. Our model accounts for $n = 11$ groups, describing the 11 Local Public Health Agencies (LPHA) in the state, see Fig. 3. Model parameters summarized in Table I. We computed an optimal perturbation using (12), and by conducting a search over all edges for perturbations of smallest norm, obtaining $\Delta = -0.0398 e_6 e_6^\mathsf{T}$. Using (16), we obtain $\Delta B = -0.0362 e_6 e_6^\mathsf{T}$, which yields $\mathcal{R}_0(B + \Delta B, \beta, \gamma) = 1.000$. Simulation results are illustrated in Fig 4. With respect to the case of no interventions (solid lines), the simulated model subject to interventions (dotted lines) experiences a decrease in the number of infections over time. For the controlled model, the number of infections tends to zero asymptotically, thus confirming that a perturbation successfully reduces the basic reproduction number and forces the system to a disease-free equilibrium.



(a) (b)

Fig. 3: (a) Partitioning of the state of Colorado, USA, according to its 11 LPHA regions. (b) Graph describing inter-regional infection intensities (see Table I for the adjacency matrix of this graph).

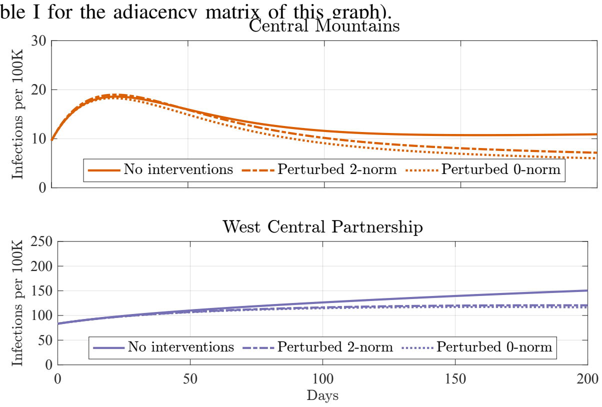


Fig. 4: Simulation results of the SIS model (15) for two selected regions in Fig. 3. The simulation shows the evolution of state trajectories for three cases: no interventions (solid lines), optimal intervention computed using Theorem 5.1 (dotted lines), and optimal interventions obtained by minimizing the 2-norm as in [15] (dashed lines). Without interventions, the states converge to an endemic equilibrium; with interventions, the states converge to a disease-free equilibrium.

VII. Conclusions

We studied the resilience of a dynamic network model against perturbations of the communication edges. Despite the problems of this class are known to be hard in the literature, in this paper we tackled the problem using a graph-theoretic approach, which makes our formulation tractable. Our results show that only edges that appear in cycles of the graph topology, if perturbed, could deteriorate the asymptotic properties of the nominal system. Overall, our results provide for the first time a graph-theoretic characterization of the set of edges that play a role in determining network resilience. Our findings come with assumptions whose relaxation could inspire several future works, including accounting for cases where A is constrained to be a a graph Laplacian, accounting for nonlinear dynamical models, considering other measures of resilience other than the 0-norm, and the derivation of countermeasures to robustify the vulnerable edges.

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