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The Role of Diameter in the Controllability of Complex Networks

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Abstract— **This paper studies the controllability degree of complex networks as a function of the network diameter and weights. We quantify the controllability degree of a network with the worst-case control energy to drive the network to an arbitrary state. We show that certain networks, including acyclic networks, are difficult to control whenever their diameter is a sublinear function of the network size, as the control energy grows exponentially with the network cardinality when the number of control nodes remains constant. Conversely, we show that certain anisotropic networks where the diameter depends linearly on the network cardinality are easy to control, as the control energy is bounded independently of the network cardinality and number of control nodes. We conjecture that the network diameter is a key topological property determining the controllability degree of a network.**

I. INTRODUCTION

Real-world networks exhibit complex topological features and dynamics across diverse engineering applications and natural systems. The ability to control and reconfigure complex networks via external controls is fundamental to guarantee reliable and efficient network functionalities. Despite important advances in the theory of control of dynamical systems, a thorough characterization of the intricate relation between relevant dynamical properties and different topological features is a long standing problem, which limits our ability to control and design complex network systems.

In this work we quantify the effort to control large networks with respect to their topology. In particular, we measure the degree of controllability of a network based on the energy required by a group of nodes to control the network to a desired state. For our metric, we investigate how the controllability degree depends on the network diameter, and on the choice of network weights [1], [2] . We find that networks with long diameter and *anisotropic* weights are easier to control than networks with short diameter or *isotropic* weights. We define weights to be isotropic if they allow a (control) signal to propagate equally in all directions, and to be anisotropic otherwise [3]. Together with our prior results [3], our findings constitute a counterintuitive exception to recent works showing that complex networks are difficult to control from few nodes [4], [5].

Related work The classic controllability notion for dynamical systems, e.g., see [1], [6], has found renewed interest in the context of complex networks, where classic methods are often inapplicable due to the system dimension, and where a graph-inspired understanding of controllability rather than a matrix-theoretical one is preferable. In [7] controllability of complex networks is addressed from a graph-theoretic perspective by employing tools from structured control theory [6]. As discussed in [4], the approach to controllability undertaken in [7] has several limitations, including the fact that the presented results are *generic* [8], and do not account for the network weights. As we also show in this work, networks with the same interconnection structure but different weights may exhibit drastically different controllability properties.

The binary notion of controllability proposed in [2] and adopted in most work studying controllability of complex networks, including [9], [10], [11], [12], does not characterize the difficulty of the control task. In practice, although a network may be controllable by any single node, the actual control input may not be implementable due to actuator constraints and limitations. A quantitative approach to network controllability has recently been adopted in [3], [4], [13], [14], [15], [16], among others, where it is shown how certain controllability metrics depend upon the network cardinality, location of the control nodes, and network weights. We continue the work along these directions by studying the relation between a notion of controllability degree of a network, its diameter, and its edge weights.

Contributions The contribution of this paper is twofold. First, we study the relation between the controllability degree of a network and its diameter. We prove that, under a technical condition on the edge weights, network topologies where the diameter is a sublinear function of the network cardinality are difficult to control, as the control energy grows exponentially with the network cardinality for a fixed number of control nodes (Section III). Second, we prove that certain networks whose diameter is a linear function of the network cardinality are easy to control (Section IV). In particular, we show that ring networks, in addition to line networks and certain grids analyzed in [3], can be efficiently controlled by a single node when the network weights satisfy a given inequality, as the control energy remains bounded independently of the network cardinality. As a final contribution, we provide evidence that acyclic random networks are difficult to control when the network cardinality increases and the number of control nodes remains bounded.

II. PROBLEM SETUP AND PRELIMINARY NOTIONS

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let $A \in \mathbb{R}^{n \times n}$ be the weighted adjacency matrix of $\mathcal{G}$, where $A = [A_{ij}]$, with $A_{ij} = 0$ if

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$(i,j) \notin \mathcal{E}$, and $A_{ij} \in \mathbb{R}$ otherwise. Let $\mathcal{K} = \{k_1, \dots, k_m\} \subseteq \{1, \dots, n\}$ be the set of control nodes, and consider the following network dynamic

$$x(t+1) = Ax(t) + B_{\mathcal{K}} u_{\mathcal{K}}(t),$$

where $B_{\mathcal{K}} \in \mathbb{R}^{n \times m}$ is the network input matrix defined as

$$B_{\mathcal{K}} := \begin{bmatrix} e_{k_1} & \dots & e_{k_m} \end{bmatrix},$$

and e_i is the i-th canonical vector of dimension n. Let $\mathcal{C}_{\mathcal{K},T}$ and $\mathcal{W}_{\mathcal{K},T}$ be the T-steps controllability matrix and Gramian, respectively, where [1]

$$\mathcal{C}_{\mathcal{K},T} := \begin{bmatrix} B_{\mathcal{K}} & AB_{\mathcal{K}} & \cdots & A^{T-1}B_{\mathcal{K}} \end{bmatrix}, \text{ and}$$

$$\mathcal{W}_{\mathcal{K},T} := \mathcal{C}_{\mathcal{K},T}\mathcal{C}_{\mathcal{K},T}^{\mathsf{T}}.$$

In this paper we study the relation between the smallest eigenvalue of the controllability Gramian, namely $\lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},T})$, as a function of the network weights and graph topological properties. We employ a classic result in systems theory that relates the control energy with the smallest eigenvalue of the controllability Gramian. Let the network be controllable in T steps, and let x_{f} be the desired final state at time T, with $\|x_{\text{f}}\|_2 = 1$. Define the energy of the control input $u_{\mathcal{K}}$ as

$$\mathrm{E}(u_{\mathcal{K}}, T) := \|u_{\mathcal{K}}\|_{2,T}^2 = \sum_{\tau=0}^{T-1} \|u_{\mathcal{K}}(\tau)\|_2^2, \tag{1}$$

where T is the control horizon. The unique control input that steers the network state from $x(0) = 0$ to $x(T) = x_{\text{f}}$ with minimum energy is

$$u_{\mathcal{K}}^*(x_{\text{f}}, t) := B_{\mathcal{K}}^{\mathsf{T}}(A^{\mathsf{T}})^{T-t-1}\mathcal{W}_{\mathcal{K},T}^{-1}\, x_{\text{f}},$$

with $t \in \{0, \dots, T-1\}$. It can be verified that

$$\mathrm{E}(u_{\mathcal{K}}^*, T) = \sum_{\tau=0}^{T-1} \|u_{\mathcal{K}}^*(x_{\text{f}}, \tau)\|_2^2 = x_{\text{f}}^{\mathsf{T}}\mathcal{W}_{\mathcal{K},T}^{-1}x_{\text{f}} \leq \lambda_{\text{min}}^{-1}(\mathcal{W}_{\mathcal{K},T}), \tag{2}$$

where equality is achieved whenever x_{f} is an eigenvector of $\mathcal{W}_{\mathcal{K},T}$ associated with $\lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},T})$. As recently shown, e.g., see [3], [4], [14], the controllability Gramian and the underlying network structure are related in a nontrivial and at times counterintuitive fashion. In particular, in Section III we show that networks with *short* diameter (we will make this statement precise in Theorem 3.1) are difficult to control with few control nodes. On the other hand, as we prove in Section IV, certain networks with *long* diameter may be easy to control with a single control node, depending on the network weights (see Section IV). As formally discussed in [3], we say that a network is easy to control if, for a fixed number of control nodes, the control energy (1) is bounded – equivalently, the smallest eigenvalue $\lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},T})$ admits a positive lower bound – independently of the network cardinality and target state. On the other hand, a network is difficult to control if the control energy grows unbounded whenever the network cardinality increases and the number of control nodes is fixed.

The following graph-theoretic notions will be used throughout the paper [17]. For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a path of length p is a sequence of p vertices $\{i_1, i_2, \dots, i_p\} \subseteq \mathcal{V}$, such that subsequent vertices are connected, that is $(i_k, i_{k+1}) \in \mathcal{E}$ for all $k \in \{1, \dots, p-1\}$. The diameter of a graph equals the length of the longest shortest path between any two distinct vertices. A cycle is a path where the first and last vertices coincide. Finally, a cycle is simple if it has no repeated vertices, except for the first and last ones.

III. Network controllability and diameter: difficult-to-control networks

In this section we characterize the relation between the controllability degree of a network system and its diameter, for certain classes of networks. We start with the following preliminary definition.

Definition 1: ***(DSS matrix)*** A matrix $M \in \mathbb{R}^{n \times n}$ is *diagonally similar to a symmetric matrix (DSS)* if there exists a nonsingular diagonal matrix $D = \text{diag}\{d_1, \dots, d_n\}$ satisfying $DMD^{-1} = \left(DMD^{-1}\right)^{\mathsf{T}} = D^{-1}M^{\mathsf{T}}D$. □

For a matrix $M \in \mathbb{R}^{n \times n}$, let $\text{cond}(M)$ denote its condition number defined as $\text{cond}(M) = \|M\|\|M^{-1}\|$. The next theorem characterizes the controllability degree of DSS networks.

Theorem 3.1: ***(Controllability of DSS networks)*** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted, undirected and connected graph. Let $A \in \mathbb{R}^{n \times n}$ be the weighted adjacency matrix of $\mathcal{G}$. Assume that A is DSS with diagonal matrix $D \in \mathbb{R}^{n \times n}$. Let $\mathcal{W}_{\mathcal{K},T}$ be the controllability Gramian associated with the pair $(A, B_{\mathcal{K}})$. Then, for all $T \in \mathbb{N}_{>0}$ and for all $\mu \in [\lambda_{\text{min}}(A), 1)$ it holds

$$\lambda_{\text{min}}(\mathcal{W}_{\mathcal{K},T}) \leq \text{cond}^2(D)\, \frac{\mu^{2(\lceil \frac{n_\mu}{|\mathcal{K}|} \rceil - 1)}}{1 - \mu^2},$$

where $n_\mu = |\{\lambda \,:\, \lambda \in \text{spec}(A), |\lambda| \leq \mu\}|$.

Proof: Let $A = D^{-1}SD$, where S is a symmetric matrix. Let W be an eigenvector matrix of S, and notice that $V = D^{-1}W$ is an eigenvector matrix of A. Since S is symmetric, there exists an eigenvector matrix W with $\text{cond}(W) = 1$. Thus,

$$\begin{aligned}\text{cond}(V) &= \|D^{-1}W\|\|W^{-1}D\| \leq \|D\|\|D^{-1}\|\|W\|\|W^{-}1\| \\ &\leq \|D\|^2\|D^{-1}\|\text{cond}(W) \leq \|D\|\|D^{-1}\|.\end{aligned}$$

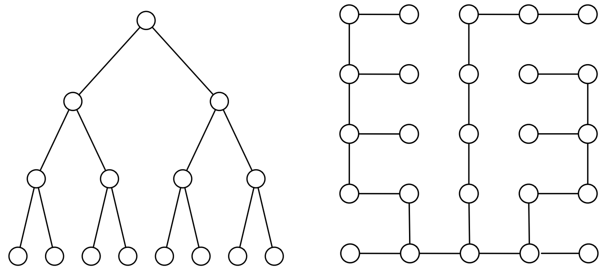
The statement then follows from [4, Theorem 3.1]. ■

As an important consequence of Theorem 3.1, if n_μ grows with the network cardinality for some μ, the number of control nodes is fixed, and the condition number $\text{cond}(D)$ admits an upper bound independent of the network cardinality, then the control energy must grow exponentially with the network cardinality. Thus, the network is difficult to control. We are now ready to characterize the relation between the controllability Gramian of a DSS network and its diameter. For notational convenience, we define

$$A_{\text{max}} := \max_{(i,j)\in\mathcal{E}} |A_{ij}|, \text{ and } A_{\text{min}} := \min_{(i,j)\in\mathcal{E}} |A_{ij}|$$

and let $\text{sgn}(\cdot)$ denote the sign function.

Theorem 3.2: ***(DSS matrix and cycles)*** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted, undirected and connected graph. Let $A \in$



(a) Binary tree (b) Grid spanning tree

Fig. 1. Fig. 1(a) and Fig. 1(b) show a binary tree and a spanning tree of a two dimensional grid network, respectively. Since the diameter of the binary tree (resp. two dimensional grid network) is $O(\log(n))$ (resp. $O(\sqrt{n})$), and due to Corollary 3.3 and Theorem 3.1, both networks are difficult to control, for a fixed number of control nodes.

$\mathbb{R}^{n\times n}$ be the weighted adjacency matrix of $\mathcal{G}$, and assume that $\mathrm{sgn}(A_{ij}) = \mathrm{sgn}(A_{ji})$ for all $(i, j) \in \mathcal{E}$. The following statements are equivalent:

(i) the matrix A is DSS, and

(ii) for all simple cycles $\{i_1, i_2, \dots, i_s, i_1\}$ of $\mathcal{G}$, it holds

$$\prod_{h=1}^{s} A_{i_h,i_{h+1}} = \prod_{h=1}^{s} A_{i_{h+1},i_h}.$$

Moreover, if A is DSS, then for all $T \in \mathbb{N}_{>0}$ and for all $\mu \in [\lambda_{\min}(A), 1)$ it holds

$$\lambda_{\min}(\mathcal{W}_{\mathcal{K},T}) \leq \left(\frac{A_{\max}}{A_{\min}}\right)^{2\cdot \mathrm{diam}(\mathcal{G})} \frac{\mu^{2\left(\lceil \frac{n_\mu}{|\mathcal{K}|}\rceil -1\right)}}{1-\mu^2},$$

where $n_\mu = |\{\lambda \,:\, \lambda \in \mathrm{spec}(A), |\lambda| \leq \mu\}|$.

Proof: We start by showing that statement (i) implies statement (ii). Let $DAD^{-1} = D^{-1}A^{\mathsf{T}}D$, with D a non-singular diagonal matrix. Notice that $D^2A = A^{\mathsf{T}}D^2$ and, consequently, $A_{ij}d_i^2 = A_{ji}d_j^2$ for all $i, j \in \{1, \dots, n\}$. Let $\{i_1, i_2, \dots, i_s, i_1\}$ be a simple cycle of $\mathcal{G}$. We have

$$\frac{A_{i_1i_2}}{A_{i_2i_1}}\frac{A_{i_2i_3}}{A_{i_3i_2}} \cdots \frac{A_{i_si_1}}{A_{i_1i_s}} = \frac{d_{i_2}^2}{d_{i_1}^2}\frac{d_{i_3}^2}{d_{i_2}^2} \cdots \frac{d_{i_1}^2}{d_{i_s}^2} = 1,$$

which concludes the proof of statement (i).

To show that (ii) implies (i), let $\mathcal{T} = (\mathcal{V}, \mathcal{E}_\mathcal{T})$ be a breath-first spanning tree of $\mathcal{G}$ rooted at the node 1 [17]. Let $d_1 = 1$, and construct the entries d_i, with $i \in \{2, \dots, n\}$, recursively as follows. Let i be a node at distance $\ell \geq 1$ from 1 on $\mathcal{T}$, and let j be at distance $\ell - 1$ from 1 on $\mathcal{T}$, with $(i, j) \in \mathcal{E}_\mathcal{T}$. Let $d_i = d_j\sqrt{A_{ji}/A_{ij}}$. Notice that $\sqrt{A_{ji}/A_{ij}}$ is a real number because $\mathrm{sgn}(A_{ij}) = \mathrm{sgn}(A_{ji})$. By construction, $A_{ij}d_i^2 = A_{ji}d_j^2$ for all $(i, j) \in \mathcal{E}_\mathcal{T}$. Let $(i, j) \notin \mathcal{E}_\mathcal{T}$, and let $\{i_1, \dots, i_s\}$ be a path on $\mathcal{T}$ from $i_1 = i$ to $i_s = j$. Due to assumption (ii) on the cycles of $\mathcal{G}$ we have

$$\begin{aligned} 1 &= \frac{A_{i_1i_2}}{A_{i_2i_1}}\frac{A_{i_2i_3}}{A_{i_3i_2}} \cdots \frac{A_{i_{s-1}i_s}}{A_{i_si_{s-1}}}\frac{A_{i_si_1}}{A_{i_1i_s}} \\ &= \frac{d_{i_2}^2}{d_i^2}\frac{d_{i_3}^2}{d_{i_2}^2} \cdots \frac{d_j^2}{d_{i_{s-1}}^2}\frac{A_{ji}}{A_{ij}} = \frac{d_j^2}{d_i^2}\frac{A_{ji}}{A_{ij}}. \end{aligned}$$

Since $A_{ij}d_i^2 = A_{ji}d_j^2$ for all $(i, j) \in \{1, \dots, n\}$, we have that $D^2A = A^{\mathsf{T}}D^2$, and finally that (i) is equivalent to (ii).

To prove the last part of the theorem, let $A = D^{-1}SD$, where S is a symmetric matrix. Since D is diagonal, $\|D\| = d_{\max}$, and $\|D^{-1}\| = 1/d_{\min}$, where $d_{\max} = \max\{d_1, \dots, d_n\}$ and $d_{\min} = \min\{d_1, \dots, d_n\}$. Let i be at distance ℓ from the root node 1. By the above construction, it can be verified that $d_i \leq d_j\sqrt{A_{\max}/A_{\min}}$, where $(i, j) \in \mathcal{T}$, j is at distance $\ell - 1$ from the root node 1, and $A_{\max}$ (resp. $A_{\min}$) is the largest (resp. smallest) entry of A. Recall that the distance between the root node 1 and any other node is bounded by $\mathrm{diam}(\mathcal{G})$. Consequently, since $d_1 = 1$, we have

$$\begin{aligned} d_{\max} &\leq (A_{\max}/A_{\min})^{\mathrm{diam}(\mathcal{G})/2}, \\ d_{\min} &\geq (A_{\min}/A_{\max})^{\mathrm{diam}(\mathcal{G})/2}, \end{aligned}$$

and, consequently,

$$\mathrm{cond}(D) = d_{\max}/d_{\min} \leq (A_{\max}/A_{\min})^{\mathrm{diam}(\mathcal{G})}.$$

The claimed statement follows from Theorem 3.1. ■

We next analyze the case of acyclic networks.

Corollary 3.3: ***(Acyclic network)*** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected, connected and acyclic graph. Let $A \in \mathbb{R}^{n\times n}$ be the weighted adjacency matrix of $\mathcal{G}$. Then, A is DSS.

Proof: The statement follows from Theorem 3.2, and the fact that an acyclic graph contains no simple cycles. ■

Theorem 3.2 and Corollary 3.3 have important consequences for the controllability of networks with bounded weights, that is, when $A_{\max} \leq \overline{A}$ and $A_{\min} \geq \underline{A}$, for some $\overline{A}, \underline{A} \in \mathbb{R}$. In fact, for networks of increasing cardinality, if the network diameter is a sublinear function of the network cardinality, and $n(\mu)$ grows linearly with n for some $\mu < 1$, then the network is difficult to control, as the control energy grows exponentially with the network cardinality, for a fixed number of control nodes. This result is in accordance with recent findings [4], [5], and it provides a novel insight into the different reasons that render a network system difficult to control. The case of networks whose diameter is a linear function of the network cardinality is considered in the next section.

To conclude this section, we show that an accurate choice of (anisotropic) network weights is necessary to guarantee that a network is easy to control.[1] In fact, from the proof of Theorem 3.2 and Corollary 3.3 we observe that an acyclic network with random entries that are independent and identically distributed is difficult to control. Indeed, notice from Theorem 3.1 that, for some constant $c \in \mathbb{R}$,

$$\mathbf{E}[\log \lambda_{\min}(\mathcal{W}_{\mathcal{K},T})] \leq c + 2\mathbf{E}[\log \mathrm{cond}(D)] - \tfrac{\mathbf{E}[n(\mu)]}{|\mathcal{K}|} \log \mu^{-1},$$

where $\mathbf{E}$ denotes the expectation operator. Thus, to show that a network with random weights is difficult to control it is sufficient to prove that $\mathbf{E}[\log \lambda_{\min}(\mathcal{W}_{\mathcal{K},T})]$ tends to $-\infty$.

Typically, for random matrices of increasing dimension, the eigenvalues distribution is described by a probability

[1] A network is easy to control if the control energy admits an upper bound independent of the network cardinality, for a fixed number of control nodes [3].

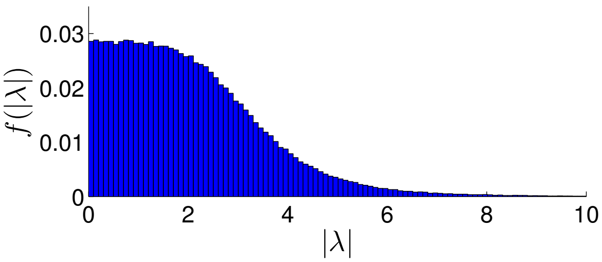


Fig. 2. Eigenvalues density function of a tridiagonal matrix (1000 rows) with i.i.d. entries. The diagonal entries are normally distributed with zero mean and unit variance. The off-diagonal entries are exponentials of normally distributed random variables with zero mean and unit variance.

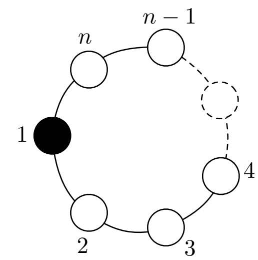


Fig. 3. Ring network topology. This figure presents the control node marked in black and nodes enumeration choice as described in (3).

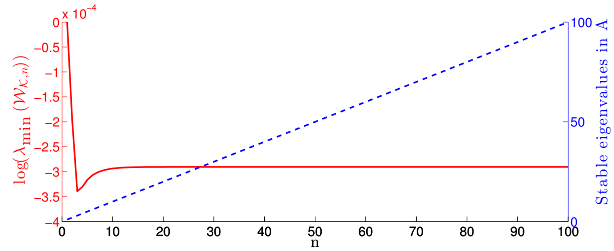


Fig. 4. This figure shows the controllability degree (red) of the ring network, and the number of stable eigenvalues (dashed blue), as a function of the cardinality n. Network weights are chosen as $d = 0.01$, $b = 0.01$, $c = 1.12$, $\bar{b} = \frac{1}{2^n}$, $\bar{c} = 0.01$, which satisfy the conditions in Theorem 4.1.

density function; see Fig. 2 for a numerical example. If the support of this asymptotic density function intersects the open interval $]0, \mu[$, then $\mathbf{E}[n(\mu)]$ tends to be proportional to n, that is, $\mathbf{E}[n(\mu)] \simeq \gamma n$, where γ is the integral of the density function in the interval $]-\mu, \mu[$. In these cases, in order to prove that an acyclic network is difficult to control, it suffices to show that $\mathbf{E}[\log \operatorname{cond}(D)]$ is sublinear.

From the proof of Theorem 3.2, the entries of the diagonal matrix D are recursively constructed based on a spanning tree $\mathcal{T}$ rooted at node 1. In particular, let $\{i_1, i_2, \ldots, i_s\}$ be a path on $\mathcal{T}$, with $i_1 = 1$. We have

$$d_{i_k} = \sqrt{\frac{A_{i_1 i_2} \cdots A_{i_{k-1} i_k}}{A_{i_2 i_1} \cdots A_{i_k i_{k-1}}}},$$

from which we obtain

$$\ell_{i_k} = \sum_{h=1}^{k-1} x_h,$$

where $\ell_{i_h} := \log d_{i_h}$ and

$$x_{i_h} := \frac{1}{2}(\log A_{i_h i_{h+1}} - \log A_{i_{h+1} i_h}).$$

Observe that x_{i_h} are independent and identically distributed random variables with zero mean. Consequently, ℓ_{i_k} represents a random walk. Let

$$\ell_{\max} := \max\{\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_{k-1}}\}, \text{ and}$$
$$\ell_{\min} := \min\{\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_{k-1}}\}.$$

From [18] we know that $\mathbf{E}[\ell_{\max} - \ell_{\min}]$ grows as the square root of the path length, which is always less that n. Hence,

$$\mathbf{E}[\log \operatorname{cond}(D)] = \mathbf{E}[\ell_{\max} - \ell_{\min}]$$

grows at most as $\sqrt{n}$, which is a sublinear function. Together with Theorem 3.2, this discussion shows that acyclic networks with random weights are difficult to control.

IV. Network Controllability and Diameter: Easy-to-Control Networks

In the previous section we showed that, if the network diameter is a sublinear function of the network cardinality, then the network is difficult to control. In this section, instead, we present an example of network that is easy to control by a single node and whose diameter is a linear function of the network cardinality. In particular, we focus on ring networks (see Fig. 3), and we remark that similar conclusions have been drawn for line and grid networks in [3]. Let

$$A := \begin{bmatrix} d_1 & b_1 & & \bar{b} \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ \bar{c} & & c_{n-1} & d_n \end{bmatrix}, \tag{3}$$

be the weighted adjacency matrix of a ring network, where all weights d_i, c_i, b_i, $\bar{c}$, and $\bar{b}$ are real and positive. In what follows we consider the case where the set of control nodes is a singleton and, without affecting generality, we let $\mathcal{K} = \{1\}$ and $\mathcal{W}_{\mathcal{K},T} = \mathcal{W}_T$. Moreover, we assume that the network weights satisfy $b_i = b$, $c_i = c$, and $d_i = d$ for some $b, c, d \in \mathbb{R}_{>0}$ and for all indices i. Our methods can be extended to more general edge weights at the cost of a more involved notation. In the following theorem we relate the controllability degree of a ring network with the entries of its adjacency matrix.

Theorem 4.1: ***(Controllability degree of ring networks)*** Consider a ring network with adjacency matrix as in (3), with $b_i = b$, $c_i = c$, and $d_i = d$ for some $b, c, d \in \mathbb{R}_{>0}$ and for all indices i. Assume that $b \leq c$, $\bar{c} < c$, and

$$\frac{1}{|c - \bar{c}|}\left(1 + b + d + \frac{\bar{b}\bar{c}}{c}\right) < 1.$$

Then, for all $T \geq n$, it holds

$$\lambda_{\min}(\mathcal{W}_T) \geq \Psi,$$

where $\Psi \in \mathbb{R}_{>0}$ is a constant dependent only on the network parameters b, c, d, $\bar{b}$, and $\bar{c}$.

A numerical validation of Theorem 4.1 is presented in Fig. 4, where we notice that, for the given choice of network weights, the smallest eigenvalue of the controllability Gramian remains bounded as the network cardinality increases. It should be noticed that the number of stable eigenvalues of the network matrix increases linearly with the cardinality. Thus, the fact the controllability degree is independent of the network size is due to the anisotropic nature of the edge weights, and not to the instability of the network system (see the discussion in Section II and [4, Theorem 3.1]). The proof of Theorem 4.1 builds on two preliminary results, which we now present.

Lemma 4.2: ***(Similarity transformation of ring networks)*** Let $A \in \mathbb{R}^{n\times n}$ be the weighted adjacency matrix of a ring network as in (3). Assume that n is even, and the weights satisfy $b_i = b$, $c_i = c$, and $d_i = d$ for some $b, c, d \in \mathbb{R}_{>0}$ and for all indices i. Define the invertible matrix $N \in \mathbb{R}^{n\times n}$ as

$$N = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & g_{\frac{n}{2}-1} & \ddots & & \\ & \iddots & & & & \\ 0 & g_1 & & & & 1 \end{bmatrix}, \tag{4}$$

where $g_i = c^{-i}(b^{i-1}\bar{c})$, and

$$N_{ij} = \begin{cases} 1, & \text{if } i = j, \\ g_{j-1}, & \text{if } i = n-j+2 \text{ and } j = 2, \dots, \frac{n}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\tilde{A} = NAN^{-1} = \begin{bmatrix} d & h_1 & & & & & h_2 \\ c & \ddots & b & & & & \\ & \ddots & & \ddots & & & \\ & & h_3 & & & & \\ & & & h_4 & & \ddots & \\ & & & & \ddots & \ddots & b \\ & & & & & c & d \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned} h_1 &= b + \bar{c}\, c^{-1}\bar{b}, \\ h_2 &= \bar{b}, \\ h_3 &= c + \bar{c}\ (b/c)^{(n/2-1)}, \\ h_4 &= c - \bar{c}\ (b/c)^{(n/2-1)}, \end{aligned}$$

and

$$\tilde{A}_{ij} = \begin{cases} d, & \text{if } i = j, \\ b, & \text{if } i = j-1, \\ c, & \text{if } i = j+1, \\ h_1, & \text{if } i = 1 \text{ and } j = 2, \\ h_2, & \text{if } i = 1 \text{ and } j = n, \\ h_3, & \text{if } i = n/2+1 \text{ and } j = n/2, \\ h_4, & \text{if } i = n/2+2 \text{ and } j = n/2+1. \end{cases}$$

Notice that (A, B), and $(\tilde{A}, B)$ are algebraically equivalent systems [1]. Lemma 4.2 can be verified by inspection; a detailed proof is omitted here in the interest of space. We now characterize the controllability matrix of the dynamical system described by the matrices in Lemma 4.2.

Lemma 4.3: ***(Controllability matrix of line and ring networks)*** Let $\tilde{A}$ be as in (5), and let $B = [1\,0\,\cdots\,0]^\mathsf{T}$. Let $\bar{A}$ be the associated tridiagonal matrix defined as $\bar{A}_{ij} = \tilde{A}_{ij}$ if $|i-j| \le 1$, and $\bar{A}_{ij} = 0$ otherwise. Let $\tilde{\mathcal{C}}_n$ (resp. $\bar{\mathcal{C}}_n$) be the n-steps controllability matrix of $(\tilde{A}, B)$ (resp. $(\bar{A}, B)$). Then,

$$\tilde{\mathcal{C}}_n := \begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} \\ 0 & e_{2,2} & \cdots & e_{2,n} \\ \vdots & \ddots & \ddots & e_{n-1,n} \\ 0 & \cdots & 0 & e_{n,n} \end{bmatrix},$$

where $e_{1,1} = 1$, and

$$e_{i,j} = \tilde{A}_{i,i-1}e_{i-1,j-1} + \tilde{A}_{i,i}e_{i,j-1} + \tilde{A}_{i,i+1}e_{i+1,j-1}.$$

Moreover,

$$\tilde{\mathcal{C}}_n = \bar{\mathcal{C}}_n.$$

Proof: From definition of Controllability matrix we have

$$\tilde{\mathcal{C}}_n(:, j) = \tilde{A}\tilde{\mathcal{C}}_n(:, j-1).$$

Then, from the structure of the matrix (5) we obtain

$$\tilde{\mathcal{C}}_n(i, j) = \sum_{k=i-1}^{i+1} A(i, k)\tilde{\mathcal{C}}_n(i, j-1),$$

which yields the triangular structure of $\tilde{\mathcal{C}}_n$. To conclude the proof notice that, since $\tilde{\mathcal{C}}_n$ is an upper triangular matrix, the nonzero element $\tilde{A}_{1n}$ does not appear in $\tilde{\mathcal{C}}_n$. Thus, by using an induction argument, $B = \tilde{\mathcal{C}}_n(:, 1) = \bar{\mathcal{C}}_n(:, 1)$, and moreover

$$\tilde{\mathcal{C}}_n(:, j) = \tilde{A}\tilde{\mathcal{C}}_n(:, j-1) = \bar{A}\bar{\mathcal{C}}_n(:, j-1) = \bar{\mathcal{C}}_n(:, j).$$

■

As an important consequence of Lemma 4.3, because the value h_2 does not appear in the controllability matrix of the equivalent pair $(\tilde{A}, B)$, the controllability of ring networks can be studied with the tools developed in [3] for the case of line networks. We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1: Let A be the weighted adjacency matrix of a ring network with $b_i = b$, $c_i = c$, and $d_i = d$, and let $B = [1\,0\,\cdots\,0]^\mathsf{T}$. Let N be as in (4), and consider the similarity transformation given by N^{-1}. In particular, $\tilde{A} = NAN^{-1}$ and $\tilde{B} = NB = B$. It can be verified that

$$\tilde{\mathcal{C}}_n = N\mathcal{C}_n, \text{ and } \tilde{\mathcal{W}}_n = \tilde{\mathcal{C}}_n\tilde{\mathcal{C}}_n^\mathsf{T} = N\mathcal{W}_nN^\mathsf{T},$$

where $\mathcal{C}_n$ and $\mathcal{W}_n$ (resp. $\tilde{\mathcal{C}}_n$ and $\tilde{\mathcal{W}}_n$) denote the n-steps controllability matrix and Gramian of (A, B) (resp. $(\tilde{A}, \tilde{B})$).

Let $x_\mathrm{f} \in \mathbb{R}^n$, with $\|x_\mathrm{f}\|_2 = 1$ be the network target state, and let $u^*(x_\mathrm{f}, t)$ be the minimum-energy input to drive the network to the state x_f in n steps. Notice that, for all $T \geq n$,

$$\begin{aligned}
\max_{\|x_\mathrm{f}\|_2=1} \mathrm{E}(u^*(x_\mathrm{f}, t), T) &\leq \max_{\|x_\mathrm{f}\|_2=1} \mathrm{E}(u^*(x_\mathrm{f}, t), n) \\
&= \lambda_{\text{min}}^{-1}(\mathcal{W}_n) \\
&= \max_{\|x_\mathrm{f}\|_2=1} x_f^\mathsf{T}\mathcal{W}_n^{-1}x_f \\
&= \max_{\|x_\mathrm{f}\|_2=1} x_f^\mathsf{T}N^\mathsf{T}\tilde{\mathcal{W}}_n^{-1}Nx_f \\
&= \max_{\|y\|_2=\|N\|_2} y^\mathsf{T}\tilde{\mathcal{W}}_n^{-1}y \\
&\leq \|N\|_2^2\lambda_{\text{min}}^{-1}(\tilde{\mathcal{W}}_n).
\end{aligned}$$

Thus,

$$\lambda_{\text{min}}(\mathcal{W}_n) \geq \frac{\lambda_{\text{min}}(\tilde{\mathcal{W}}_n)}{\|N\|_2^2}. \qquad (6)$$

From Lemma 4.3 we have $\tilde{\mathcal{C}}_n = \bar{\mathcal{C}}_n$, where $\bar{\mathcal{C}}_n$ is the controllability matrix of $(\bar{A}, B)$, and $\bar{A}$ is the matrix containing the tridiagonal structure of $\tilde{A}$. Notice that $\bar{A}$ is the weighted adjacency matrix of a line network. From (6) and [3, Theorem 2.1], under the conditions stated in the theorem we obtain

$$\lambda_{\text{min}}(\mathcal{W}_n) \geq \frac{\lambda_{\text{min}}(\tilde{\mathcal{W}}_n)}{\|N\|_2^2} \geq \frac{\Psi'}{\|N\|_2^2},$$

where Ψ' is a constant depending only on the entries of $\bar{A}$. To conclude the proof notice from (4) that [19]

$$\|N\|_2 \leq \sqrt{\|N\|_1\|N\|_\infty} \leq 1 + \bar{c}/b,$$

where the last inequality follows from the fact that $b \leq c$. ■

V. Conclusion

In this paper we study the relation between the controllability degree of a network system, which we measure based on the control energy needed to drive the network to an arbitrary target state, and its diameter. We show that, for a class of networks where the diameter is a sublinear function of the network cardinality, the control energy depends exponentially on the ratio between the network cardinality and the number of control nodes. Thus, in these cases the control energy grows unbounded for networks with increasing dimension and fixed number of control nodes. This analysis includes to the case of acyclic networks with deterministic or random weights. Conversely, we show that there exist networks where the control energy admits an upper bound independent of the network cardinality and target state. These networks feature a diameter that is a linear function of the network cardinality, and a set of weights that guarantee an anisotropic propagation of the control signal.

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