Event-Triggered Feedback Optimization of LTI Systems with Applications to Pandemic Control

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Abstract

Feedback optimization is an increasingly utilized control paradigm to regulate dynamical systems to solutions of optimization problems. Existing methods for feedback optimization heavily rely on the availability of continuous sensing, communication, and actuation resources, which may be impractical in a number of real-world systems. In this context, this paper proposes a framework for event-triggered feedback optimization, where discrete events are used to decide when to update control inputs and feedback signals in a resource-efficient fashion; in particular, triggering events are governed by values of internal state variables. We provide a systematic method to design the optimization-based controllers as well as explicit triggering laws that lessen the communication burden, while preserving the asymptotic stability properties of the interconnection between system and controllers. As an application example, we consider the problem of controlling the intensity of non-pharmaceutical interventions in epidemic control. Here, the proposed framework is especially relevant since an infrequent testing of the population against the disease can reduce the economic cost related to testing, while resource-efficient control updates lead to less frequent updates of the active pandemic mitigation measures.

1 Introduction

In this paper, we study the problem of designing a feedback controller for a linear dynamical system, to steer its inputs and outputs to the solution of a convex optimization problem. In particular, the problem is parametrized by an unknown disturbances affecting the system. The controller features an even-based mechanism that automatically selects – based on the value of internal state variables, control inputs, and outputs – the instants at which the system outputs are sampled and the control inputs are updated, to bypass the need for continuous communication and sensing.

The need for event-based controllers and event-based sensing is central to many engineering and scientific applications where actuation, sensing, and communication must be accomplished in a resource-aware fashion. As an example, limited sensing and actuation emerges in epidemiological applications, where policymakers seek to mitigate the outbreak of a certain disease by repeatedly intensifying or lifting pandemic mitigation measures, such as social distancing, mask-wearing mandates, and short lockdowns [1]. In the case of the COVID-19 pandemic, policymakers worldwide have faced the following central questions such as: *when should mitigation measures be intensified or lifted based on hospitalizations or treatments capacities?* And, *how frequently should the population be tested to estimate the epidemic state?* While frequent policy updates allow policymakers to quickly respond to dynamically-changing conditions, it is practically unfeasible to enforce mitigation measures that are updated too frequently (e.g. daily). Similarly, frequent testing allows to accurately estimate the epidemic state and to keep track of daily cases while, on the other hand, frequent testing is economically impractical. Along the same vein, event-trigger mechanisms allows for communication-efficient sensing and control frameworks in, e.g., power grids [2] and transportation [3].

Related works.

The use of optimization algorithms as feedback controllers for dynamical systems has received significant attention during the last decade. A necessarily incomplete list of recent works includes [4,5] for linear systems, [6,7] for nonlinear systems, [8] for simultaneous system stabilization and optimization, [9] for constrained optimization, [7,10] for the use of accelerated optimization algorithms, and [11] for stochastic optimization and a data-enabled implementation of the controller. Especially relevant to this paper is the recent work [12], where the authors study a general framework to optimize dynamical systems where communication is sampled in a periodic fashion. Yet, the aforementioned work employs periodic transmission protocols, which may be resource-inefficient for many real-world systems; differently, in this work we focus on cases where the controller must automatically select the time instants at which communication is needed based on the state of internal vari-

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ables. Such departure is tackled here by adopting the framework of Hybrid Dynamical Systems to model and design the even-based logics, and our analysis is accomplished by leveraging a hibrid version of the invariance principle [13]. Because of this, the vast body of literature on event-based control is also related to this work. Generally, there are two event-based approaches that are commonly adopted in practice [14]: event-triggered control and self-triggered control. In the former, an event – such as a controller update or a data transmission – is triggered only when certain internal state variables meet a certain condition. This condition should be tested at each state or output, thus requiring continuous monitoring of the system or controller [15]. Self-triggered control, on the other hand, determines the next sampling time and transmission once a sampled measurement is received, and thus does not require to continuously sample the states [16] and can be implemented by leveraging model-free representations [17].

Contributions. In this work, we propose control framework that combines techniques from feedback optimization and event-triggered control. The contribution of this paper is threefold. First, we formulate a steady-state optimal control problem where communication from the controller to the plant and from the plant to the controller can occur only at discrete time instants. We focus on two cases: optimization where the control inputs are transmitted only at discrete time instants and optimization where the feedback signals are sensed only at discrete time instants. Second, by using the powerful framework of Hybrid Dynamical Systems [18], we derive explicit, state-dependent, triggering conditions that ensure that the controller asymptotically regulates the plant to a first-order optimizers of the control objective. Third, we study the applicability of the framework on a network Susceptible-Infected-Susceptible epidemic model tailored for the state of Colorado, USA. Overall, our numerical results suggest that, in order to regulate the infectious state to a desired threshold with arbitrary precision, it is sufficient to update restrictions and sample test the population, on average, every 9 days.

Organization. Section 2 provides relevant definitions, Section 3 formalizes the studied problem, Sections 4 and 5 contain the main technical contributions, and Section 6 applies the framework to an epidemic control problem. Finally, Section 7 concludes the paper.

2 Preliminaries

In this section, we introduce here some basic terminology and notions. Given a symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\underline{\lambda}(M)$ and $\bar{\lambda}(M)$ denote its smallest and largest eigenvalue, respectively; $M \succ 0$ indicates that M is positive definite. For $u \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $(x, u) \in \mathbb{R}^{n+m}$ denotes vector concatenation. We denote by $\|u\|$ the Euclidean norm of u and by $u^\top$ its transposition. Given a nonempty compact set $\mathcal{A} \subset \mathbb{R}^n$, $|u|_{\mathcal{A}} := \inf_{z\in\mathcal{A}} \|z-u\|$ denotes the point-to-set distance. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, $\gamma(0) = 0$, and strictly increasing; it is of class $\mathcal{K}_\infty$ if in addition it is unbounded.

We will use the framework of Hybrid Dynamical Systems (HDS) to analyze event-based logics. A HDS with state $\xi \in \mathbb{R}^n$ is described by

$$\xi \in C,\ \dot{\xi} \in F(\xi), \qquad \xi \in D,\ \xi^+ \in G(\xi), \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ are the flow and jump maps, respectively, whereas $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed sets denoting, respectively, the flow and jump sets. The vector fields F and G are assumed to be continuous on C and D, respectively. The continuity of F and G, together with the closedness of C and D ensure that the system is *well-posed* [18, Ch. 6]. Solutions to (1) are parametrized by two time indices: a continuous index $t \in \mathbb{R}_{\geq 0}$ that increases continuously whenever the system flows in C as $\dot{\xi}(t, j) := \frac{d}{dt}\xi(t, j) \in F(\xi(t, j))$; and a discrete index $j \in \mathbb{Z}_{\geq 0}$ that increases by one whenever the system jumps in D as $\xi^+ := \xi(t, j + 1) \in G(\xi(t, j))$. Solutions to (1) are defined on hybrid time-domains [18, Def. 2.3], namely, subsets of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ defined as the union of intervals $[t_j, t_{j+1}] \times \{j\}$, with $0 = t_0 \leq t_1 \leq \ldots$, and where the last interval can be closed or open on the right. We denote by $\mathrm{dom}(\xi)$ the domain of ξ. We say that the solutions to (1) have a *uniform semiglobal dwell-time outside* $\mathcal{A}$, where $\mathcal{A} \subseteq \mathbb{R}^n$ is strongly forward pre-invariant for (1), if for any $\Delta \geq 0$ there exists $\tau(\Delta) > 0$ such that for any solution ξ to (1) with $|\xi(0, 0)|_{\mathcal{A}} \leq \Delta$ and any $(s, i), (t, j) \in \mathrm{dom}(\xi)$ with $s + i \leq t + j$:

$$\xi(t, j) \notin \mathcal{A} \Rightarrow j - 1 \leq (t - s)/\tau(\Delta) + 1.$$

We will use the following standard notions [18, Def. 3.6].

Definition 2.1 *The closed set $\mathcal{A} \subset \mathbb{R}^n$ is uniformly globally pre-asymptotically stable (UGpAS) for (1) if:*

(i) [Uniform global stability] There exists $\alpha \in \mathcal{K}_\infty$ such that for any solution ξ to (1), $|\xi(t, j)|_{\mathcal{A}} \leq \alpha(|\xi(0, 0)|_{\mathcal{A}})$ for all $(t, j) \in dom(\xi)$.
(ii) [Uniform global pre-attractivity] For each $\varepsilon, r > 0$, there exists $T > 0$ such that for any solution ξ to (1) with $|\xi(0, 0)|_{\mathcal{A}} \leq r$, $(t, j) \in dom(\xi)$ and $t + j \geq T$ imply $|\xi(t, j)|_{\mathcal{A}} \leq \varepsilon$.

Set $\mathcal{A}$ is uniformly globally asymptotically stable (UGAS) when, in addition, the maximal solutions to (1) are complete.

3 Problem Formulation

We consider continuous-time systems with dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew, \tag{2}$$

where $t \in \mathbb{R}_{\geq 0}$ denotes time, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the state, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the control decision, and $w \in \mathbb{R}^q$ (constant) describes disturbances affecting the state dynamics. In the remainder, we will drop time indices for notational simplicity. We make the following assumptions on the plant (2).

Assumption 1 *The system* (2) *is controllable. Moreover, the matrix A is Hurwitz stable, i.e., there exists $P \succ 0$ such that $A^\mathsf{T} P + PA^\mathsf{T} = -I$.* □

Assumption 1 guarantees asymptotic stability of the unique equilibrium point of (2), which is given by:

$$x = \underbrace{-A^{-1}B}_{:=G} u + \underbrace{-A^{-1}E}_{:=H} w, \tag{3}$$

for any given $u \in \mathbb{R}^m, w \in \mathbb{R}^q$. In cases where the underlying plant does not satisfy the above stability property, it can be first stabilized using standard feedback control techniques; in these case, (2) models the pre-stabilized system.

We are interested in driving the input and the state of (2) to the solutions of the following optimization problem:

$$\begin{aligned} (u^*, x^*) \in \arg \min_{u^\mathrm{d}, x^\mathrm{d}} \quad & \Phi(u^\mathrm{d}, x^\mathrm{d}), \\ \text{s.t.} \quad & x^\mathrm{d} = Gu^\mathrm{d} + Hw, \end{aligned} \tag{4}$$

where $\Phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is a loss function that models costs associated with the inputs and states, and w is a *parameter* of the optimization problem describing the disturbances in (2).

Remark 1 *Problem* (4) *is an equilibrium-selection problem, where the objective is to control* (2) *to an equilibrium point* (u^*, x^*) *that minimizes the cost* Φ. *Notice that, even though we assume that the equilibria of* (2) *are stable, the plant has equilibrium points that are parametrized by the system inputs (see* (3)*). Hence, the goal of* (4) *is to regulate the plant to an equilibrium point that is optimal (in the sense that it minimizes* Φ*), despite not being able to measure* w. □

One challenge associated with the control objective (4) is that solutions to the optimization problem cannot be computed explicitly because w is unknown, and thus the control problem can be interpreted as a *regulation problem to unknown (but optimal) equilibrium point.*

In what follows, we define the set

$$\mathcal{A} := \{(u^*, x^*) \ : \ (u^*, x^*) \text{ is a first-order optimizer of (4)}\},$$

and we define the matrix $\Pi^\mathsf{T} := \begin{bmatrix} I_m & G^\mathsf{T} \end{bmatrix}$.

We make the following assumption on the set $\mathcal{A}$ and on the cost Φ.

Assumption 2 *The set $\mathcal{A}$ is nonempty and closed. Moreover, the function $\Phi(\cdot, \cdot)$ is convex and there exists $\ell \in \mathbb{R}_{>0}$:*

$$\|\Pi^\mathsf{T}(\nabla\Phi(u, x) - \nabla\Phi(u, x'))\| \leq \ell\|x - x'\|, \tag{5}$$

for all $x, x' \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. □

The control objective (4), has been studied in e.g. [7, 9, 10]. In these works, the authors have proposed low-gain gradient-type controllers of the form:

$$\dot{u} = -\eta\Pi^\mathsf{T}\nabla\Phi(u, x), \tag{6}$$

where $\eta > 0$ is a scalar gradient gain to be tuned. The controller (6) is of the form of a gradient-flow algorithm – which is the natural approach to solve optimization problems of the form (4) – yet modified by replacing the true gradient $\Pi^\mathsf{T}\nabla\Phi(u, Gu + Hw)$ with an approximate gradient evaluated at the instantaneous system state $\Pi^\mathsf{T}\nabla\Phi(u, x)$, thus making the iteration (6) independent of the unknown disturbance w.

Remark 2 *As shown in [9], the dynamics* (6) *can be adapted to account for convex constraints on the input of the form $u \in \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^m$ is a closed and convex set. In these cases,* (6) *shall be modified to $\dot{u} = P_\mathcal{U}(u - \eta\Pi^\mathsf{T}\nabla\Phi(u, x)) - u$, where $P_\mathcal{U}(\cdot)$ denotes the Euclidean projection. Notice that the above projected dynamics admit solutions that are continuously differentiable [9].* □

Departing from these works, the focus of this paper is to solve the control task (4) where the control inputs and system states are, respectively, applied and sensed *only when needed.* We will consider two scenarios for this problem. First, (see Fig. 1(a)) we consider cases where the control signals are computed in a continuous fashion but they are released only at (non-necessarily periodic) discrete time instants $0 \leq t_0 < t_1 < \cdots < \infty$. Between two transmissions, the plant no longer has access to u, instead, it has access to a sampled version of it, denoted by $\hat{u}$. Here, $\hat{u}$ is generated by an arbitrary holding function implemented locally (at the plant):

$$\dot{\hat{u}} = F_\mathrm{u}(\hat{u}, x). \tag{7}$$

Second, (see Fig. 1(b)) we consider cases where the feedback signal x is released only at discrete time instants $0 \leq t_0 < t_1 < \cdots < \infty$. Between two transmissions, the controller no longer has access to x, instead, it has access only to its sampled version $\hat{x}$, generated by an arbitrary holding function implemented locally (at the controller):

$$\dot{\hat{x}} = F_\mathrm{x}(u, \hat{x}) \tag{8}$$

In both cases, we will study the following problem.

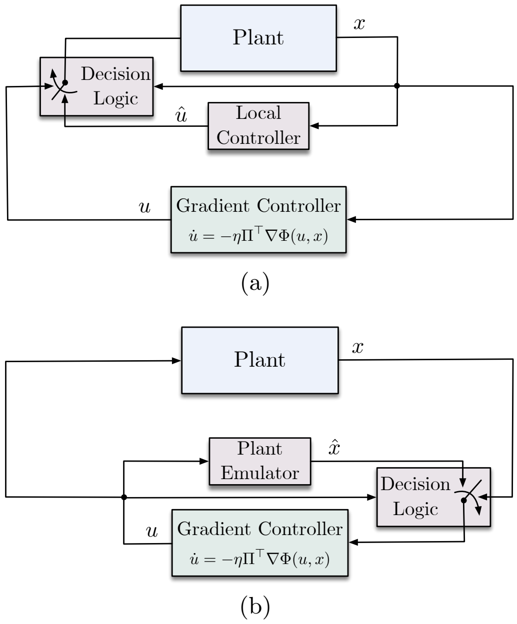


Fig. 1. (a) Feedback optimization with event-triggered input: control signals u are released only at discrete time-instants and between transmissions the plant has access to a sampled version $\hat{u}$ generated by a holding function. (b) Feedback optimization with event-triggered state measures: feedback signals x are released only at discrete time-instants and between transmissions the controller has access to a sampled version $\hat{x}$ generated by a plant emulator.

Problem 1 *Design a logic that selects the time instants $\{t_0, t_1, \dots\}$ as well as the parameter η to guarantee that any solution of (2) converges asymptotically to the set of first-order optimizers $\mathcal{A}$.* $\square$

4 Feedback optimization with event-triggered control input

In this section, we study Problem 1 in cases where the control signal u is transmitted only at discrete time instants (cf. Fig. 1(a)). To this end, we will rewrite the control system (2), (6), and (7) as a HDS with state $\xi = (x, u, e)$ and

$$\xi \in C, \begin{cases} \dot{x} = Ax + B(u+e) + Ew, \\ \dot{u} = -\eta \Pi^\top \nabla\Phi(u,x), \\ \dot{e} = F_{\mathrm{e}}(u,x,e), \end{cases} \qquad \xi \in D, \begin{cases} x^+ = x, \\ u^+ = u, \\ e^+ = 0, \end{cases} \tag{9}$$

where $e = \hat{u} - u \in \mathbb{R}^m$ models the sampling-induced error, which is reset to $e = 0$ at each jump and has continuous dynamics $F_{\mathrm{e}}(u,x,e) = F_{\mathrm{u}}(e+u,x) - \dot{u}$, and $C \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ and $D \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ are, respectively, the flow and jump sets, which are parameters to be designed.

In the following result, we provide a choice for C and D that guarantees uniform global asymptotic stability of $\mathcal{A}$.

Theorem 4.1 *Consider the dynamics (9), and let the flow and jump sets be, respectively,*

$$\begin{aligned} C &= \{\xi \,:\, \|e\| \leq \frac{\sigma}{2\|PB\|}\|A^{-1}\dot{x}\|\}, \\ D &= \{\xi \,:\, \|e\| \geq \frac{\sigma}{2\|PB\|}\|A^{-1}\dot{x}\|\}, \end{aligned} \tag{10}$$

where $\sigma \in (0,1)$ is a free parameter. Then, there exists $\bar{\eta} \in \mathbb{R}_{>0}$ such that for all $\eta \in (0, \bar{\eta})$ the set $\mathcal{A}$ is UGAS for (9). Moreover, the solutions to (9) have a uniform semiglobal dwell-time outside $\mathcal{A}$. $\square$

PROOF. The proof of UGAS relies on showing that (9) with flow and jump sets (10) satisfies the assumptions of the Hybrid Invariance Principle [13], and is inspired from [18, Thm. 5.19]. First, we notice that because the sets C and D are closed and the flow and jump maps $F(\xi)$ and $G(\xi)$ are continuous, the system (9) satisfies the *hybrid basic conditions* [13]. Next, we consider the change of variables $\tilde{x} := x - Gu - Hw$, which shifts the equilibrium point of (2) to the origin. In the new variables, flow and jump sets of (9) read, respectively, as

$$F(\tilde{\xi}) = \begin{pmatrix} A\tilde{x} + Be + A^{-1}B\dot{u} \\ -\eta\Pi^\top \nabla\Phi(u, \tilde{x} + Gu + Hw) \\ F_{\mathrm{e}}(u, \tilde{x} + Gu + Hw, e) \end{pmatrix}, \quad G(\tilde{\xi}) = \begin{pmatrix} \tilde{x} \\ u \\ 0 \end{pmatrix},$$

where $\tilde{\xi} = (\tilde{x}, u, e)$. In what follows, we will denote by $\tilde{\mathcal{A}} = \{0\} \times \{u^*\} \times \mathbb{R}^m$. Inspired by singular perturbation reasonings [19], consider the Lyapunov function candidate on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ with respect to $\tilde{\mathcal{A}}$ given by:

$$U(\tilde{x}, u) =:= \frac{1}{\eta}V(u) + \frac{1}{\eta}W(\tilde{x}), \tag{11}$$

for each $\tilde{\xi} \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, where

$$\begin{aligned} V(u) &= \Phi(u, Gu + Hw) - \Phi(u^*, Gu^* + Hw), \\ W(\tilde{x}) &= \tilde{x}^\top P \tilde{x}. \end{aligned} \tag{12}$$

We next bound the time-derivative of $V(u)$:

$$\begin{aligned} \frac{d}{dt}V(u) &= [\Pi^\top \nabla\Phi(u, Gu + Hw) - \underbrace{\Pi^\top \nabla\Phi(u^*, Gu^* + Hw)}_{=0}]^\top \dot{u} \\ &\leq -\eta\|\Pi^\top \nabla\Phi(u, \tilde{x} + Gu + Hw)\|^2 \\ &\quad + \eta\ell\|\Pi^\top \nabla\Phi(u, \tilde{x} + Gu + Hw)\|\|\tilde{x}\|, \end{aligned} \tag{13}$$

where the inequality follows from Assumption 2. Next, we bound the time-derivative of $W(\tilde{x})$:

$$\frac{d}{dt}W(\tilde{x}) = \tilde{x}(A^\mathsf{T}P + PA)\tilde{x} + 2\tilde{x}^\mathsf{T}PBe + 2\tilde{x}^\mathsf{T}PA^{-1}B\dot{u} \tag{14}$$
$$\begin{aligned} &\leq -\|\tilde{x}\|^2 + 2\|PB\|\|\tilde{x}\|\|e\| \\ &\quad + 2\eta\|PA^{-1}B\|\|\tilde{x}\|\|\Pi^\mathsf{T}\nabla\Phi(u,\tilde{x}+Gu+Hw)\| \\ &\leq -(1-\sigma)\|\tilde{x}\|^2 \\ &\quad + 2\eta\|PA^{-1}B\|\|\tilde{x}\|\|\Pi^\mathsf{T}\nabla\Phi(u,\tilde{x}+Gu+Hw)\|, \end{aligned}$$

where $\sigma \in (0,1)$ is as in the assumptions and the last inequality holds when $-\sigma\|\tilde{x}\|^2 + 2\|PB\|\|\tilde{x}\|\|e\| \leq 0$ or, equivalently, $\|e\| \leq \frac{\sigma}{2\|PB\|}\|\tilde{x}\|$, which is guaranteed by the choice of flow set (10) (notice that $\tilde{x} = A^{-1}\dot{x}$). By combining (13)-(14), we have $\frac{d}{dt}U(\tilde{x},u) \leq -z^\mathsf{T}\Lambda z$ with

$$z = (\|\Pi^\mathsf{T}\nabla\Phi(u,\tilde{x}+Gu+Hw)\|, \|\tilde{x}\|), \tag{15}$$

and

$$\Lambda = \begin{bmatrix} 1 & -\frac{1}{2}(\ell + 2\|PA^{-1}B\|) \\ -\frac{1}{2}(\ell + 2\|PA^{-1}B\|) & \frac{1-\sigma}{\eta} \end{bmatrix}.$$

By noting that $\Lambda \succ 0$ when $\frac{1-\sigma}{\eta} > \frac{1}{4}(\ell + 2\|PA^{-1}B\|)$, we conclude that for any $\eta \in (0,\bar{\eta})$, $\bar{\eta} = \frac{4(1-\sigma)}{\ell+2\|PA^{-1}B\|}$, the function $U(\tilde{x},u)$ satisfies

$$\frac{d}{dt}U(\tilde{x},u) \leq -\underline{\lambda}(\Lambda)\|z\|^2, \quad \forall\, \tilde{\xi} = (\tilde{x},u,e) \in C. \tag{16}$$

Hence, since z is positive definite with respect to the optimal points, we conclude that $U(\tilde{x},u)$ strictly decreases during flows away from $\tilde{\mathcal{A}}$. At jumps, since $G(\cdot)$ does not change the state components $(\tilde{x},u)$, it follows that

$$U(G(\tilde{\xi})) = U(\tilde{\xi}), \qquad \forall\, \tilde{\xi} = (\tilde{x},u,e) \in D.$$

Thus, by the above two conclusions and because (10) satisfies the basic hybrid conditions, the Hybrid Lyapunov Theorem [13, Thm. 3.19, item 1] guarantees that the set $\tilde{\mathcal{A}}$ is stable. To show pre-attractivity, notice that $U^{-1}(\{0,u^*\}) \subset \tilde{\mathcal{A}}$ and $\Delta U^{-1}(\{0,u^*\}) = D$. Thus, since (9) satisfies the hybrid basic conditions, the Hybrid Invariance Principle [13, Thm. 3.23, item 1] guarantees that every precompact solution to (9) converges to the largest invariant subset of

$$U^{-1}(r) \cap (\tilde{\mathcal{A}} \cup (D \cap G(D))), \tag{17}$$

for some $r \geq 0$. To determine $D \cap G(D)$, notice that $\xi \in D$ implies $\frac{d}{dt}U(\tilde{x},u) \geq -\underline{\lambda}(\Lambda)\|z\|^2$ and, by combination with (16), this is only possible when $\xi \in \tilde{\mathcal{A}}$. Then, (17) can only hold when $r = 0$ or, equivalently, when $\xi \in \tilde{\mathcal{A}}$. To conclude pre-attractivity of $\tilde{\mathcal{A}}$, we are left to show that there exists $\delta > 0$ such that every maximal solution $\tilde{\xi} = (\tilde{x},u,e)$ with $\|\tilde{\xi}(0,0)\|_{\tilde{\mathcal{A}}} < \delta$ is bounded, i.e., $\|(\tilde{x},u)\| < \epsilon$ for some $\epsilon > 0$. Since $U(\tilde{x},u)$ is positive definite with respect to $\mathcal{A}$, its level sets are compact and thus stability of $\tilde{\mathcal{A}}$ implies that $\|(\tilde{x},u)\| < \epsilon$. This concludes the proof of UGAS. Finally, existence of a semiglobal dwell-time outside $\mathcal{A}$ follows by application of [13, Prop. 5.11] by noting that (9) satisfies the hybrid basic conditions and from the fact that every maximal solution is bounded.

Theorem 4.1 provides an explicit triggering logic and an existence-type characterization for the gain η that guarantee convergence of the plant and controller trajectories to the critical points of the optimization problem (4). We notice that the triggering condition (10) requires to evaluate the sampling error e with respect to the time-derivative of the state x. The latter can be computed robustly in practice by using, e.g., passive differentiators. Moreover, the theorem guarantees that there exists a positive lower bound on the time between consecutive events; this feature is critical to many physical implementations of the controller as it guarantees the existence of non-Zeno solutions, which would otherwise require arbitrarily fast computations.

5 Feedback optimization with event-triggered state measures

The focus of this section is on cases where the feedback signal x is transmitted only at a sequence of discrete time instants (cf. Fig. 1(b)). To this aim, we rewrite the control system (2), (6), and (8) as a HDS with state $\xi = (x,u,e)$ and

$$\xi \in C, \begin{cases} \dot{x} = Ax + Bu + Ew, \\ \dot{u} = -\eta\Pi^\mathsf{T}\nabla\Phi(u,x+e), \\ \dot{e} = F_\mathrm{e}(u,x,e), \end{cases} \qquad \xi \in D, \begin{cases} x^+ = x, \\ u^+ = u, \\ e^+ = 0, \end{cases} \tag{18}$$

where $e = \hat{x} - x \in \mathbb{R}^n$ models the sampling-induced error, which is reset to $e = 0$ at each jump and has continuous dynamics $F_\mathrm{e}(u,x,e) = F_\mathrm{x}(u,e+x) - \dot{x}$, and $C \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ are, respectively, the flow and jump sets, which are to be designed.

In the following result, we provide a choice for the sets C, D that guarantees uniform global asymptotic stability of the set $\mathcal{A}$ in the presence of sampled state measures.

Theorem 5.1 *Consider the dynamics* (18)*, and let the flow and jump sets be, respectively,*

$$\begin{aligned} C &= \{\xi \ : \ \|e\| \leq \max\{\alpha\|\Pi^\mathsf{T}\nabla\Phi(u,x)\|, \beta\|A^{-1}\dot{x}\|\}\}, \\ D &= \{\xi \ : \ \|e\| \geq \max\{\alpha\|\Pi^\mathsf{T}\nabla\Phi(u,x)\|, \beta\|A^{-1}\dot{x}\|\}\}, \end{aligned} \tag{19}$$

where $\alpha = \frac{\sigma}{\ell}$, $\beta = \frac{\sigma}{\eta\ell(\ell+2\|PA^{-1}B\|)}$, and $\sigma \in (0,1)$ is a free parameter. Then, there exists $\bar{\eta} \in \mathbb{R}_{>0}$ such that for all $\eta \in (0,\bar{\eta})$ the set $\mathcal{A}$ is UGAS for (18). *Moreover, the solutions to* (18) *have a uniform semiglobal dwell-time outside $\mathcal{A}$.* □

PROOF. The proof of this result follows similar steps as the proof of Theorem 4.1, suitable modified to account for errors in the output signal. More precisely, we will show that (18) with flow and jump sets (19) satisfies the assumptions of the Hybrid Invariance Principle [13]. Consider the change of variables $\tilde{x} = x - Gu - Hw$, which shifts the equilibrium point of (2) to the origin. In the new variables, flow and jump sets of (9) read, respectively, as

$$F(\tilde{\xi}) = \begin{pmatrix} A\tilde{x} + A^{-1}B\dot{u} \\ -\eta\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw + e) \\ F_{\mathrm{e}}(u, \tilde{x} + Gu + Hw + e, e) \end{pmatrix}, \quad G(\tilde{\xi}) = \begin{pmatrix} \tilde{x} \\ u \\ 0 \end{pmatrix},$$

where $\tilde{\xi} = (\tilde{x}, u, e)$. Next, we let $\tilde{\mathcal{A}} = \{0\}\times\{u^*\}\times\mathbb{R}^m$ and we consider the Lyapunov function candidate defined in (11)-(12). The time-derivative of $V(u)$ reads as:

$$\begin{aligned}\frac{d}{dt}V(u) &= [\Pi^{\mathsf{T}}\nabla\Phi(u, Gu + Hw) - \underbrace{\Pi^{\mathsf{T}}\nabla\Phi(u^*, Gu^* + Hw)]}_{=0}{}^{\mathsf{T}}\dot{u} \\ &= [\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)]^{\mathsf{T}}\dot{u} \\ &\quad + [\Pi^{\mathsf{T}}\nabla\Phi(u, Gu + Hw) - \Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)]^{\mathsf{T}}\dot{u}.\end{aligned}$$

The first term satisfies:

$$\begin{aligned}[\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)]^{\mathsf{T}}\dot{u} &\leq -\eta\|\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)\|^2 \\ &\quad + \eta\ell\|\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)\|\|e\|,\end{aligned}$$

and the second term satisfies:

$$\begin{aligned}&[\Pi^{\mathsf{T}}\nabla\Phi(u, Gu + Hw) - \Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)]^{\mathsf{T}}\dot{u} \\ &\quad \leq \eta\ell\|\tilde{x}\|(\|\nabla\Phi(u, \tilde{x} + Gu + Hw)\| + \ell\|e\|),\end{aligned}$$

where the inequality follows from Assumption 2. By combining the above estimates, we conclude that:

$$\begin{aligned}\frac{d}{dt}V(u) &\leq -\eta\|\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)\|^2 \\ &\quad + \eta\ell\|\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)\|\|e\| \\ &\quad + \eta\ell\|\tilde{x}\|\|\nabla\Phi(u, \tilde{x} + Gu + Hw)\| + \eta\ell^2\|\tilde{x}\|\|e\|.\end{aligned} \tag{20}$$

Next, we bound the time-derivative of $W(\tilde{x})$:

$$\begin{aligned}\frac{d}{dt}W(\tilde{x}) &= \tilde{x}(A^{\mathsf{T}}P + PA)\tilde{x} + 2\tilde{x}^{\mathsf{T}}PA^{-1}B\dot{u} \\ &\leq -\|\tilde{x}\|^2 + 2\|PA^{-1}B\|\|\tilde{x}\|\|\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw + e)\| \\ &\leq -\|\tilde{x}\|^2 + 2\eta\|PA^{-1}B\|\|\tilde{x}\|\|\Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)\| \\ &\quad + 2\eta\ell\|PA^{-1}B\|\|\tilde{x}\|\|e\|.\end{aligned} \tag{21}$$

By combining (20)-(21) and by denoting in compact form $g(u, \tilde{x}) = \Pi^{\mathsf{T}}\nabla\Phi(u, \tilde{x} + Gu + Hw)$:

$$\begin{aligned}\frac{d}{dt}U(\tilde{x}, u) &\leq -\|g(u, \tilde{x})\|^2 + \ell\|g(u, \tilde{x})\|\|e\| + \ell\|\tilde{x}\|\|g(u, \tilde{x})\| \\ &\quad + \ell^2\|\tilde{x}\|\|e\| - \frac{1}{\eta}\|\tilde{x}\|^2 + 2\|PA^{-1}B\|\|\tilde{x}\|\|g(u, \tilde{x})\| \\ &\quad + 2\ell\|PA^{-1}B\|\|\tilde{x}\|\|e\| \\ &\leq -(1-\sigma)\|g(u, \tilde{x})\|^2 + \ell\|\tilde{x}\|\|g(u, \tilde{x})\| \\ &\quad - \frac{1-\sigma}{\eta}\|\tilde{x}\|^2 + 2\|PA^{-1}B\|\|\tilde{x}\|\|g(u, \tilde{x})\|,\end{aligned}$$

where $\sigma \in (0,1)$ is as in the assumptions and the second inequality is guaranteed to hold when

$$\begin{aligned}-\sigma\|g(u, \tilde{x})\|^2 + \ell\|g(u, \tilde{x})\|\|e\| &\leq 0, \\ -\sigma\|\tilde{x}\|^2 + \eta\ell^2\|\tilde{x}\|\|e\| + 2\eta\ell\|PA^{-1}B\|\|\tilde{x}\|\|e\| &\leq 0,\end{aligned}$$

which are satisfied by the choice of flow set (19). Hence, by combining the above estimates and by adopting the notation (15), we conclude that $U(\tilde{x}, u)$ strictly decreases during flows away from $\tilde{\mathcal{A}}$, namely

$$\frac{d}{dt}U(\tilde{x}, u) \leq -z^{\mathsf{T}}\Lambda z < 0, \quad \forall\xi = (\tilde{x}, u, e) \in C \setminus \tilde{\mathcal{A}},$$

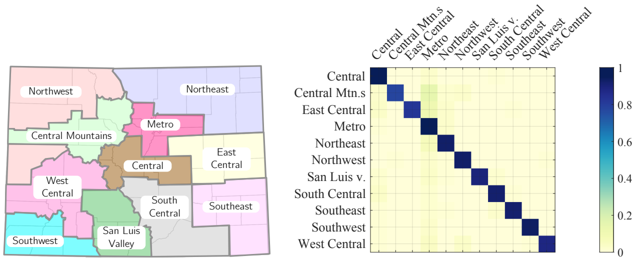
where

$$\Lambda = \begin{bmatrix} 1-\sigma & -\frac{1}{2}(\ell + 2\|PA^{-1}B\|) \\ -\frac{1}{2}(\ell + 2\|PA^{-1}B\|) & \frac{1-\sigma}{\eta} \end{bmatrix}.$$

By noting that $\Lambda \succ 0$ if and only if $\frac{(1-\sigma)^2}{\eta} > \frac{1}{4}(\ell + 2\|PA^{-1}B\|)$, we conclude that for any $\eta \in (0,\bar{\eta})$, where $\bar{\eta} = \frac{4(1-\sigma)^2}{\ell+2\|PA^{-1}B\|}$, the function $U(\tilde{x}, u)$ strictly decreases during flows. At jumps, since $G(\cdot)$ does not change the state components $(\tilde{x}, u)$, it follows that

$$U(G(\tilde{\xi})) = U(\tilde{\xi}), \qquad \forall\, \tilde{\xi} = (\tilde{x}, u, e) \in D.$$

Thus, by the above two conclusions and because (10) satisfies the basic hybrid conditions, the Hybrid Lyapunov Theorem [13, Thm. 3.19, item 1] guarantees that the set $\tilde{\mathcal{A}}$ is stable. Finally, the proof of pre-attractivity and of



(a) (b)

Fig. 2. (a) Geographical partitioning of the state of Colorado, USA, according to its 11 LPHA regions. (b) Illustration of the matrix of interactions between regions (corresponding to matrix A in (23)).

existence of a semiglobal dwell-time follow directly by iretaing the steps in the proof of Theorem 4.1.

Theorem 5.1 provides a triggering logic and an existence-type characterization for η that guarantee convergence of the plant-controller trajectories to the critical points of the optimization problem (4) uner output sampling. We notice that the triggering condition (10) requires to evaluate the sampling error e with respect to two state-dependent quantities: (i) the time-derivative of the state x, and (ii) the norm of the gradient of the cost evaluated at the instantaneous u and x. In comparison with Theorem (4.1), condition (ii) is new and can be interpreted by noting that errors in the feedback signal originate errors in the computed control input, which is itself fed into the plant thus originating a feedback-type error. Moreover, similarly to Theorem (4.1), Theorem (5.1) guarantees that there exists a positive lower bound on the time between consecutive events, thus guaranteeing existence of Zeno solutions that are non-Zeno.

6 Applications to control of epidemics

In this section, we illustrate the applicability of the framework in controlling the infectious state of an SIS epidemic model. The adopted model is deliberately simple to illustrate the control approach, however, we remark that the techniques adopted in the sequel can be easily generalized to account for more complicated models with several states (by leveraging similar feedback linearization techniques).

6.1 Network SIS model

We consider a geographical region describing an isolated state, country, or continent, and we organize it into n sub-regions denoted by $\mathcal{V} = \{1, \dots, n\}$. To every region $k \in \mathcal{V}$, we associate two state variables to describe the epidemic state in that region: $i_k : \mathbb{R}_{\geq 0} \to [0,1]$ (fraction of infected population) and $1 - i_k := s_k : \mathbb{R}_{\geq 0} \to [0,1]$ (fraction of susceptible population). We adopt a susceptible-infected-susceptible (SIS) [20] model with constant population:

$$\begin{aligned} \frac{d}{dt} s_k &= -\beta(1-r_k) s_k \sum_{\ell \in \mathcal{V}} a_{k\ell} i_\ell + \gamma i_k, \\ \frac{d}{dt} i_k &= \beta(1-r_k) s_k \sum_{\ell \in \mathcal{V}} a_{k\ell} i_\ell - \gamma i_k + w_k, \quad k \in \mathcal{V}, \end{aligned} \tag{22}$$

where $\beta > 0$ is the transmission rate, $\gamma > 0$ is the recovery rate, $a_{k\ell} > 0$ are parameters that model the intensity of infection due to interactions between susceptible individuals residing in region k and infected individuals residing in region ℓ, $w_k \in \mathbb{R}$ is a disturbance that models unknown inflows of individuals to the system, and $r_k : \mathbb{R}_{\geq 0} \to [0,1]$ is a decision variable that describes the intensity of pandemic mitigation measures adopted by policymakers to limit disease spread. More precisely, $r_k = 0$ is interpreted as "zero restrictions" and $r_k = 1$ is interpreted as "full lockdown." In vector form, the second equation of (22) reads as:

$$\dot{x} = \beta \operatorname{diag}(1-r)(I - \operatorname{diag}(x)) A x - \operatorname{diag}(\gamma \mathbb{1}) x + w, \tag{23}$$

where $r = (r_1, \dots, r_n)$ is the vector of decision variables, $x = (i_1, \dots, i_n)$ is the vector of infected states, $w = (w_1, \dots, w_n)$ is the vector of disturbances, $A = [a_{k\ell}] \in \mathbb{R}^{n \times n}$ is the matrix of interactions, $\mathbb{1} \in \mathbb{R}^n$ is the vector of all ones, and $\operatorname{diag}(v) \in \mathbb{R}^{n \times n}$ denotes the diagonal matrix whose diagonal entries are defined by the vector $v \in \mathbb{R}^n$.

Since (23) is a nonlinear model, we choose r_k as follows:

$$r_k = 1 - \frac{u_k}{(1-i_k) \sum_{\ell \in \mathcal{V}} a_{k\ell} i_\ell}, \qquad k \in \mathcal{V}, \tag{24}$$

where $u_k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a free control signal to be designed. The control (24) is a feedback-linearizing law [19, Ch. 13], which cancels the nonlinear term $(I - \operatorname{diag}(x))Ax$ from (23). With the control (24), the network dynamics (23) simplify to a linear model:

$$\dot{x} = \underbrace{-\operatorname{diag}(\gamma \mathbb{1})}_{:=\bar{A}} x + \underbrace{\operatorname{diag}(\beta \mathbb{1})}_{:=\bar{B}} u + w. \tag{25}$$

Notice that $\bar{A}$ is Hurwitz stable (since $\gamma > 0$) and (25) is controllable (since B is diagonal and $\beta > 0$), and thus (25) is in a form that satisfies our assumptions. Accordingly, in what follows we let $\bar{G} = -\bar{A}^{-1}\bar{B}$ and $\bar{H} = -\bar{A}^{-1}$. Finally, we assume that the instantaneous infections x is measurable.

6.2 Control objective

We formalize the objective of regulating the infectious state x to a desired reference setpoint $x^{\text{ref}} \in \mathbb{R}^n$ as fol-

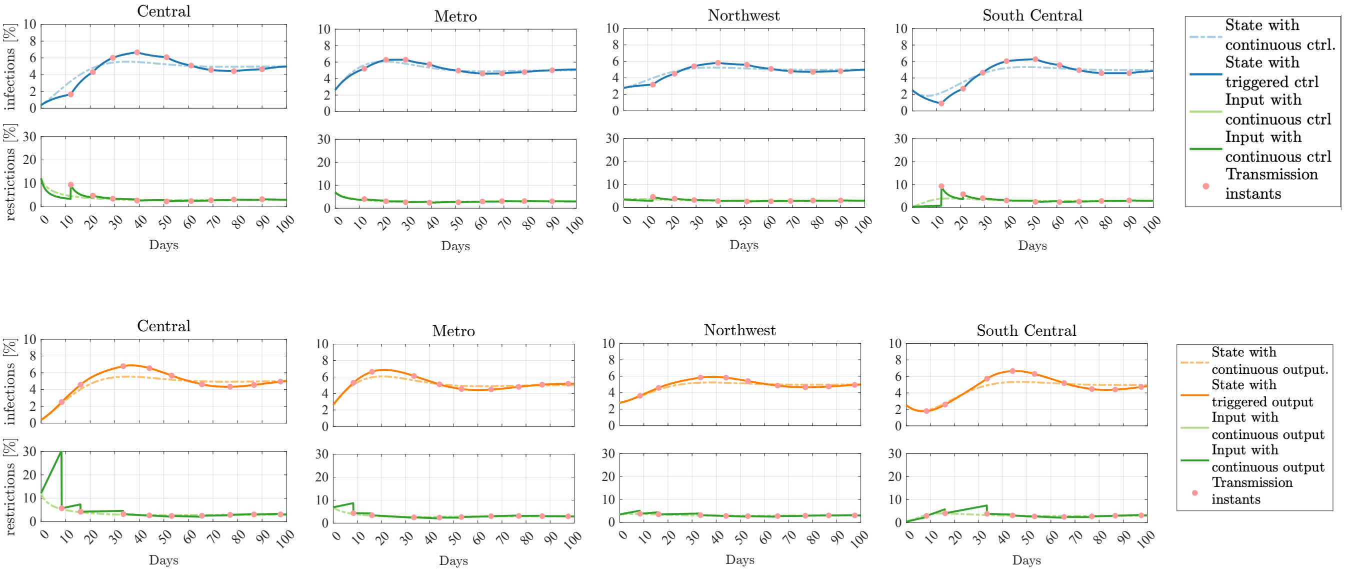


Fig. 3. Time-evolution of the infectious state i_k and pandemic mitigation measures r_k for the network SIS model (22) for four, representative, LPHA regions of the state of Colorado, USA. Model parameters: $\beta = 4, \gamma = 1/9$, and A as in Fig. 2. (Top row) Control with intermittent restriction update (see (C2)); this controller is constructed by using feedback linearization (24) and the gradient flow (9). (Bottom row) Control with intermittent testing (see (C2)); this controller is constructed using (24) and (18). In all simulations, flow and jump sets are defined with $\sigma = 1/2$. As illustrated, the event-triggering mechanism allows us to drastically reduce the frequency of updates of the restrictions and of testing, which occur, on average, every 9 days.

lows:

$$(u^*, x^*) := \arg\min_{u^{\rm d}, x^{\rm d}} \quad \|x^{\rm d} - x^{\rm ref}\|_{Q_x}^2, \qquad \text{s.t.} \quad x^{\rm d} = \bar{G}u^{\rm d} + \bar{H}w, \tag{26}$$

where $Q_x \succ 0$. In what follows, we account for two practical challenges related to the control objective (26):

(C1) **Control with intermittent restriction update:** due to the practical challenges related to enforcing quickly-changing social restriction measures, we first focus on cases where a policymaker seeks to minimize the frequency of updates of interventions (i.e., vector r).

(C2) **Control with intermittent testing:** due to the economic costs of testing, we next focus on cases where a policymaker seeks to minimize the frequency at which the state (i.e., vector x) is sensed.

The control challenges outlines above can be translated into two mathematically rigorous frameworks corresponding, respectively, to the techniques studied in Sections 4 and 5.

6.3 Simulation results for the state of Colorado, USA

We simulated the proposed feedback controller on an instance of (22) where model parameters have been interpolated from COVID-19 data for the state of Colorado, USA (model parameters have been fitted by using regional hospitalization data from the period 1/1/21-2/28/21 [21]). As illustrated in Fig. 2(a), we partitioned the state according to its $n = 11$ Local Public health Agency (LPHA) regions, and we used mobility data from cell-phone usage to estimate inter-regional couplings (see Fig. 2(b) for a description of the matrix of inter-region couplings A and [21] for data sources).

In Fig. 3 we show simulation results, over a time horizon of 100 days, in which the feedback controller seeks to regulate the infectious state to $x^{\rm ref} = 6$ infected individuals every 100 inhabitants in all regions. The top row corresponds to the case *(C1) Control with intermittent update of the restrictions*, while the second row illustrates simulations results for the case *(C2) Control with intermittent testing*. In the first case, an update of the restrictions is required on average every 9 days (corresponding to days $12, 21, 29, 39, 51, 61, 69, 78, 90$); in the second case, testing is required on average every 9 days (corresponding to days $8, 16, 33, 44, 53, 65, 77, 86, 97$). As shown, all state trajectories converge (within less than 1 person/100K inhabitants accuracy) to the desired number of infections (6 infected individuals every 100 inhabitants) within 100 days. Overall, these simulations show the applicability and benefits of framework in drastically reducing the frequency of updates of the pandemic mitigation measures and of testing.

7 Conclusions

We proposed a gradient-flow controller for steady-state optimization, which incorporates an event-based mechanism that automatically selects the time instants at which communication should occur based on the state of internal state variables. By leveraging the framework of Hybrid Dynamical Systems, we have derived sufficient conditions that guarantee stability of the closed-loop system under event-based sampling. We showed the applicability of the framework to limit infections in the context of COVID-19; overall, our results suggest that the frequency of testing and mitigation measures updates can be reduced to once every nine days without deteriorating the performance of the controller. Future research directions include extensions to more general plant models, the derivation of numerically more tractable triggering rules that overcome the need for differentiators, and incorporating model learning.

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